Nonlinear Evolutions in Banach Spaces

Existence and Qualitative Theory with Applications to Reaction-Diffusion Systems

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Introduction

The time evolution of many concrete processes from various fields like physics, chemistry, biology and engineering sciences is governed by nonlinear partial differential equations which admit an abstract formulation as evolution problems of the type

$$u' + Au \ni f(t,u) \quad \text{on } J = [0,T], \quad u(0) = u_0 \tag{0.1}$$

in an appropriate (infinite dimensional) Banach space X, where A is an m-accretive (especially nonlinear and possibly multivalued) operator and $f : D(f) \subset J \times X \to X$ is a nonlinear reacting force.

An important class of applications that falls into this scope is given by certain systems of reaction-diffusion equations, and some of the special features, concerning the settings in which (0.1) will be considered, are motivated by such applications as follows. In particular if chemically reacting systems are modeled then *diffusion* has to be taken into account. Since chemical reactions are often performed inside catalytic pellets of high porosity, diffusion will usually be nonlinear. This leads to reaction-diffusion systems of the form

$$\begin{aligned} &\frac{\partial u_k}{\partial t} = \Delta\varphi_k(u_k) + g_k(u_1,\dots,u_m) && \text{in } (0,T)\times\Omega \\ &\frac{\partial \varphi_k(u_k)}{\partial \nu} = 0 && \text{on } (0,T)\times\partial\Omega \\ &u_k(0,\cdot) = u_{0,k} && \text{in } \Omega \end{aligned} \tag{0.2}$$

for $k = 1,\dots,m$ in a bounded domain $\Omega \subset \mathbb{R}^n$, in simple cases. Here u_k denotes the concentration of a certain species, and the functions $\varphi_k : \mathbb{R} \to \mathbb{R}$ are continuous, strictly increasing with $\varphi_k(0) = 0$. This includes the case $\varphi(r) = r|r|^{\gamma-1}$ with $\gamma > 0$ encountered in the *porous medium equation.* In this situation the operators $A_k v = -\Delta\varphi_k(v)$, say with homogeneous Neumann boundary conditions, are m-accretive in $L^1(\Omega)$, hence the system (0.2) admits an abstract formulation as (0.1) in $X = L^1(\Omega)^m$. The function g in (0.2) describes the kinetics of the underlying system of chemical reactions.

In the simplest case $m = 2$ the reaction term g is typically of the type

$$g(y_1,y_2) = (-\mu h_1(y_1)h_2(y_2), -\lambda\mu h_1(y_1)h_2(y_2))$$

with $\lambda \in \mathbb{R}$, $\mu > 0$ and continuous $h_k : \mathbb{R}_+ \to \mathbb{R}$. For example, a single irreversible isothermic reaction $A + B \to P$ between two chemical species A and B is usually modeled by means of the Freundlichs kinetics, which corresponds to $h_1(s) = s^\alpha$, $h_2(s) = s^\beta$ with $\alpha,\beta > 0$ and $\lambda > 0$; here the reaction is said to have order α and β with respect to A and B, respectively. The case of a single irreversible exothermic reaction $A \to P$ of order α leads to $h_1(s) = s^\alpha$ and $h_2(s) = \exp(-\frac{E}{Rs})$ with $\alpha, E, R > 0$ and $\lambda < 0$. In this example u_1 denotes the concentration of A and u_2 means temperature.

In many cases the order of a reaction is not known from theoretical considerations but has to be determined on the basis of measurements. This often leads to fractional values and in particular $\alpha, \beta < 1$ are possible, in which case g is merely continuous. Actually, there are situations where the kinetics of a chemical reaction corresponds to order zero, which is modeled by means of the Heaviside function, i.e. $h_1(0) = 0$ and $h_1(r) = 1$ for $r > 0$ in the examples above. This is the simplest situation where reaction-diffusion systems with *discontinuous nonlinearities* arise; other examples will be given later.

Ordinary differential equations with discontinuous right-hand side need not have (absolutely continuous) solutions. A typical example is $y' = g(y)$ on $[0, 1]$ with $g(r) = -\operatorname{sgn} r$ for $r \neq 0$ and $g(0) = 1$, say. Notice that if y is a solution with initial value $y(0) = 0$, then $|y|' = y' \operatorname{sgn} y \leq 0$ a.e. implies $y(t) = 0$ on $[0, 1]$, but $y = 0$ is not a solution. A common way to overcome this difficulty is to replace the discontinuous g by an appropriate multivalued "regularization" G, which is obtained from g by filling in the gaps at points of discontinuity. Then, in the situation of (0.2), a partial differential inclusion results, the abstract formulation of which is given by

$$u' \in -Au + F(t, u) \;\text{ on } J, \;\; u(0) = u_0, \tag{0.3}$$

where $F : D(F) \subset J \times X \to 2^X \setminus \{\emptyset\}$ is a multivalued mapping.

Another feature in the kind of applications mentioned above is that only nonnegative solutions are meaningful due to the physical background. Moreover, if such a reaction-diffusion system is considered as an abstract evolution problem, the nonlinear reacting force is only defined on "thin" subsets of X, for example L^∞-bounded subsets of $L^1(\Omega)$, unless the reaction term satisfies strong and often unrealistic growth conditions. Therefore, one is often interested in solutions of problem (0.1) or (0.3) that satisfy additional constraints of the type $u(t) \in K$ for certain subsets $K \subset X$.

Consequently, we study nonlinear evolution problems of type (0.1) and (0.3) in infinite dimensional Banach spaces with emphasis on cases when the nonlinear reacting force is only defined on a closed subset of $J \times X$ and is merely continuous or Carathéodory in the single-valued case corresponding to (0.1), or satisfies a condition of upper semicontinuous type if (0.3) is considered. Based on compactness arguments we establish existence of solutions and provide further qualitative results like existence of solutions under time-dependent constraints or existence of periodic solutions, which are then applied to systems of the form (0.2) and to more complicated related models.

In the sequel, the nonlinear reacting force f (respectively F) will often be called a "perturbation" for simplicity, although this notion is not justified in the strict sense; in particular it will not be assumed that f is "small" compared to A.

The study of problem (0.3) certainly requires a combination of the different techniques that have been developed for evolution equations governed by m-accretive operators (i.e. the case $F = 0$ or $F(t, x) = \{f(t)\}$), respectively for differential inclusions in Banach spaces (i.e.

the case $A = 0$). In these special cases many aspects concerning existence and qualitative properties of solutions are well understood, and the preliminary Chapter 1 provides a compilation of known facts that are needed for the present work.

In Chapter 2 we are concerned with existence and qualitative theory in the abstract setting of (0.1) and (0.3). We start with the problem of existence of mild solutions in case that the perturbation is defined on all of $J \times \overline{D(A)}$. This is the subject of §3 where we concentrate on initial value problem (0.3) with a multivalued perturbation F with closed convex values such that $F(\cdot, x)$ admits a strongly measurable selection and $F(t, \cdot)$ is weakly upper semicontinuous; notice that (0.1) with Carathéodory f is a special case. Here $u \in C(J;X)$ is called a mild solution of (0.3) if

$$u = \mathcal{S}w \;\text{ for } w \in L^1(J;X) \text{ with } w(t) \in F(t, u(t)) \text{ a.e. on } J,$$

where $\mathcal{S}w$ denotes the mild solution of the quasi-autonomous problem

$$u' + Au \ni w(t) \;\text{ on } J, \;\; u(0) = u_0$$

associated with A. Hence u is a mild solution of (0.3) iff u is a fixed point of $G := \mathcal{S} \circ \mathrm{Sel}$, where $\mathrm{Sel} : C(J;X) \to 2^{L^1(J;X)}$ is given as

$$\mathrm{Sel}\,(u) = \{w \in L^1(J;X) : w(t) \in F(t, u(t)) \text{ a.e. on } J\}.$$

It is therefore natural to try a *fixed point approach* in order to obtain existence of solutions, but one has to be careful since (0.3) need not have a solution even in finite dimensions. This is shown by Example 3.1, where the main point is that $w_k \rightharpoonup w$ in $L^1(J;X)$ and $\mathcal{S}w_k \to u$ in $C(J;X)$ does not imply $\mathcal{S}w = u$. Consequently, all subsequent results especially depend on assumptions that guarantee certain properties of the solution operator $\mathcal{S}$, which then allow to overcome this difficulty. Of course such properties of $\mathcal{S}$ in turn rely on properties of A and X and, e.g., the problem mentioned above disappears if X^* is uniformly convex. In this situation the fixed point approach works if we find a compact convex $K \subset C(J;X)$ such that $G(K) \subset K$. Existence of a closed bounded convex K_0 with $G(K_0) \subset K_0$ follows if F has at most linear growth in x, and we then impose a certain compactness assumption on A or F to obtain the compact set K. In case the semigroup generated by $-A$ is compact such a compact convex K is easily obtained since $G(K_0)$ is relatively compact then (which gives Theorem 3.1), while additional effort is needed if the semigroup is only equicontinuous and F satisfies the compactness condition

$$\beta(F(t,B)) \le k(t)\beta(B) \;\text{ on } J \text{ with } k \in L^1(J) \text{ for bounded } B \subset \overline{D(A)},$$

where β denotes the Hausdorff-measure of noncompactness. Here the main point is to establish the estimate (Lemma 3.7)

$$\beta(\{(\mathcal{S}w_k)(t) : k \ge 1\} \le \int_0^t \beta(\{w_k(s) : k \ge 1\})\, ds,$$

which holds for $(w_k) \subset L^1(J;X)$ such that $|w_k(t)| \leq \varphi(t)$ a.e. on J with $\varphi \in L^1(J)$, if X^* is uniformly convex; without additional assumptions on X this "β-formula" may break down as shown by Example 3.2. By means of this inequality, existence of mild solutions for such perturbations "of compact type" is obtained in Theorem 3.4.

There are other relevant special cases in which the operator $\mathcal{S}$ is such that the fixed point approach works in general Banach spaces, namely if A is linear, densely defined and m-accretive, or if A is everywhere defined, continuous and accretive. Actually, existence of mild solutions can be obtained if A is a "semilinear" operator of the type $Au = A_0u + g(u)$, where $A_0 : D(A_0) \subset X \to X$ is linear, m-accretive with $\overline{D(A_0)} = X$ and $g : X \to X$ is continuous and accretive. This is done in Theorem 3.5, and an extension to time-dependent Carathéodory g is given in Theorem 3.6. The latter relies on Theorem 3.2 which in particular covers the case of dissipative Carathéodory perturbations of m-accretive operators in a general Banach space.

If constraints are present a fixed point approach is not possible, and then one main step consists in obtaining appropriate *approximate solutions.* This is the starting point of §4 where we concentrate on the single-valued case, i.e. on problem (0.1). To gain more flexibility for later applications, we allow for time-dependent constraints of type $u(t) \in K(t)$ on J where $K : J \to 2^X \setminus \{\emptyset\}$ is a given "tube", which of course requires a certain "subtangential condition" on the boundary of the graph of K. In Lemma 4.1 we use Zorn's lemma to establish existence of a carefully chosen type of approximate solutions under a necessary subtangential condition. Based on this result we obtain existence of so-called viable solutions (i.e. solutions staying in $\mathrm{gr}\,(K)$) if f is locally Lipschitz continuous (Theorem 4.1), or f is continuous and a compactness condition of one of the types mentiones above is satisfied (Theorem 4.2 and Theorem 4.3). For fixed constraints, i.e. $K(t) \equiv K$, a refinement of the type of approximate solutions is possible (Lemma 4.3), which then allows to extend the above results to the case when f is Carathéodory (Theorem 4.4 and Theorem 4.5). Since the necessary subtangential condition is hard to check in practice, we also provide stronger sufficient conditions.
By means of the techniques given in §4 it is then clear how corresponding results for multi-valued perturbations can be obtained, and a sample for ϵ-δ-usc F is Theorem 4.6.

As a consequence of the fact that compactness conditions are used to obtain existence of solutions (unless the nonlinear reacting forces are locally Lipschitz or satisfy a dissipativity condition), the set of all solutions is a compact subset of $C(J;X)$ in all the situations mentioned above. On the other hand, especially initial value problem (0.3) can have a "large" solution set and it is therefore of interest to obtain further information concerning the structure of this set. This is the first subject in §5 where we show that, in particular, the solution set of (0.3) is a compact R_δ-set (and hence connected) within the settings considered in §3; see Theorem 5.1 and Theorem 5.2.

In §5.2 we provide sufficient conditions for the existence of T-periodic solutions of (0.1) in a given closed set $K \subset X$ in case f is T-periodic and Carathéodory, where we concentrate on two different settings: X is a general Banach space and f satisfies a stronger separated subtangential condition (Theorem 5.3), or X and X^* are uniformly convex, K has nonempty interior and f satisfies an explicit subtangential condition on $K \cap D(A)$ which is a necessary condition if A is single-valued (Theorem 5.4).

We close this "abstract" chapter with a section on sums of accretive operators, where the main result (Theorem 5.5) extends the well-known fact that $A + F$ with m-accretive A and continuous and accretive $F : X \to X$ is m-accretive, to the case of upper semicontinuous F with compact convex values.

A large part of Chapter 3 is devoted to applications of the abstract theory to reaction-diffusion systems like (0.2). In §6 we first draw some immediate consequences from the results concerning existence and viability. In particular we show how common invariance techniques that are well-known in the semilinear case, like invariant rectangles in combination with quasimonotonicity or contracting rectangles, carry over to the fully nonlinear setting; see Theorems 6.1 and 6.2 as well as Proposition 6.1. We also consider systems with discontinuous nonlinearities, modeled as partial differential inclusions as explained above, where the main result (Theorem 6.3) establishes the existence of (local) nonnegative solutions under natural assumptions.
For a concrete process, say from chemical engineering, the resulting mathematical model is of course more complicated than (0.2). In §6.3 we consider a more realistic model for a standard process from heterogeneous catalysis where, in addition to nonlinear diffusion and reaction inside Ω, macroscopic convection and reaction inside the surrounding bulk phase as well as interfacial mass transport is taken into account. Since such processes are sometimes operated in a periodic manner (i.e. the feeds are varied periodically) in order to increase the performance with respect to conversion or selectivity, the problem of existence of a T-periodic solution appears naturally. Theorem 6.4 provides existence of T-periodic and stationary solutions under fairly realistic assumptions. The main ingredient for the proof of this result is a certain compactness property of the semigroup generated by the pde-part (Lemma 6.4), where the latter is similar to $\Delta\varphi(v)$ but with an additional component and a nonstandard boundary condition.

Given a system of chemical reactions it is likely that some of the reactions take place at a considerably higher rate than the remaining ones, in particular if ionic or radical reaction mechanisms are involved. Intuitively, the fast reactions will be close to their equilibrium position, which means that the vector of all concentrations evolves in the neighborhood of a certain "limiting manifold", additionally driven by the influence of feeds and slow reactions. Then a common approach in chemical engineering is to consider the fast reactions as instantaneous ones, i.e. to assume that the system is in steady state with respect to all fast reactions; in this context one speaks of "the instantaneous reaction limit". In the remaining part of Chapter 3 this approach is justified in several different situations by means of rigorous convergence results. Of course an important first step in order to solve these particular singular limit problems is to study the ideally mixed case with macroscopic convection. In this situation the reaction-diffusion system reduces to a system of ordinary differential equations with a large parameter $k > 0$ which refers to the reaction speed. In §7.1 we consider a concrete systems of concurring irreversible reactions in the ideally mixed case (Example 7.1) where the reaction term turns out to be accretive in l^1-norm. Based on perturbation results for nonlinear semigroups we obtain convergence of solutions, as k tends to infinity, to the solution of a discontinuous limiting problem.

If the fast irreversible reactions take place in the bulk phase of a more complicated two-phase process, then the limiting problem is a reaction-diffusion system with additional discontinuous nonlinearities. In the abstract formulation this leads to a nonlinear evolution problem of type (0.3), again.

In §7.2 we consider the instantaneous reaction limit for a single irreversible reaction between two diffusive species and use nonlinear semigroup theory to obtain convergence of the solutions as $k \to \infty$ to the solution of a free boundary problem (Theorem 7.1). In particular, this result shows that limiting problems with nonlinear diffusion of the type $\Delta\varphi(v)$ arise naturally in the instantaneous limit, even if the original system with finite reaction speed is semilinear.

In case of fast reversible reactions the passage to the limiting case of infinite reaction speed is more difficult in so far that the reaction part is not accretive. Here we concentrate on the ode-case corresponding to the situation when the chemical reactions are performed inside a continuously stirred tank reactor. We characterize the limiting equation and, by means of Lyapunov functions techniques, we are able to prove convergence of solutions for a general system of fast independent reactions (Theorem 8.2).

Finally, we consider a reversible reaction $A + B \rightleftharpoons P$ of diffusive species and solve the corresponding singular limit problem in the special case of equal diffusion coefficients (Theorem 8.3). Due to the strong extra assumptions this result is just a starting point for future work.

Chapter 1

Accretivity and Upper Semicontinuity

This chapter provides several basic concepts and results from nonlinear functional analysis that are fundamental to the subsequent study of evolution problems of the type described in the introduction.

§1 Accretive Evolution Equations

The purpose of this preliminary paragraph is to provide several known facts from the theory of accretive operators and nonlinear semigroups. For proofs of these facts we refer to Barbu [14], Benilan/Crandall/Pazy [17] or Miyadera [85] unless an explicit reference is given. Additional information will be given at the appropriate places later on.

1.1 Basic notation

Throughout this work the following notation will be used. X will always stand for a Banach space with norm $|\cdot|$, 2^X denotes the subsets of X and $2^X \setminus \emptyset$ is short for $2^X \setminus \{\emptyset\}$. Then $\overline{B}_r(x)$ denotes the closed ball in X with center x and radius r, $B_r(x)$ its interior and $\rho(x, A)$ the distance from x to the set $A \subset X$. For $A \subset X$ we let $\overline{A}, \mathring{A}, \partial A$ be the closure, interior and boundary of A, respectively. We also let $||A|| = \sup\{|x| : x \in A\}$. The convex hull of $A \subset X$, i.e. the smallest convex set containing A, is denoted by $\operatorname{conv} A$, while $\operatorname{span} A$ is the smallest linear subspace of X containing A. The closure of these sets are denoted as $\overline{\operatorname{conv}}\, A$ and $\overline{\operatorname{span}}\, A$, respectively. For $x \in X$, $A, B \subset X$ and $\lambda \in \mathbb{R}$ we define

$$A + \lambda B = \{a + \lambda b : a \in A, b \in B\} \quad \text{and} \quad x + B = \{x\} + B.$$

Given $A, B \in 2^X \setminus \emptyset$ we let

$$d_H(A, B) = \max\{\sup_A \rho(x, B), \sup_B \rho(x, A)\}.$$

Then d_H is a metric on the closed bounded subsets of X, the so-called *Hausdorff-metric*.

If an *operator* $A : X \to 2^X$ (an *operator* A *in* X for short) is given, we let

$$D(A) = \{x \in X : Ax \neq \emptyset\}, \quad R(A) = \bigcup_{x \in D(A)} Ax$$

and

$$\mathrm{gr}\,(A) = \{(x, y) : x \in D(A), y \in Ax\}$$

denote the *domain, range* and *graph* of A, respectively. We will occasionally identify A with its graph to simplify the notation; so, for example, $(x, y) \in A$ means $x \in D(A)$ and $y \in Ax$. An operator A is said to be *single-valued* if Ax is a singleton for every $x \in D(A)$.

Given $J = [0, a] \subset \mathbb{R}$, we let $C(J; X)$ denote the Banach space of all continuous functions $u : J \to X$ and $L^1(J; X)$ be the Banach space of all equivalence classes (with respect to equality almost everywhere) of strongly measurable, Bochner-integrable $w : J \to X$, both equipped with the usual norms, i.e.

$$|u|_0 = \max_{t \in J} |u(t)| \quad \text{and} \quad |w|_1 = \int_0^a |w(t)|dt,$$

respectively. In case $X = \mathbb{R}$ we simply write $C(J)$ and $L^1(J)$.

1.2 Accretive operators

An operator A in a real Banach space X is said to be *accretive* if

$$|x - \overline{x}| \leq |x - \overline{x} + \lambda(y - \overline{y})| \quad \text{for all } \lambda > 0,\ x, \overline{x} \in D(A), y \in Ax, \overline{y} \in A\overline{x},$$

while A is said to be ω*-accretive* (with $\omega \in \mathbb{R}$) if $A + \omega I$ is accretive, where $I : X \to X$ denotes the identity. An operator A is *dissipative* iff $-A$ is accretive.

There are equivalent definitions of accretivity that are more appropriate to check whether a given operator has this property. The subsequent formulation is based on the *duality map* $\mathcal{F} : X \to 2^{X^*} \setminus \emptyset$ (where X^* denotes the dual space of X) given by

$$\mathcal{F}(x) = \{x^* \in X^* : x^*(x) = |x|^2 = |x^*|^2\}.$$

By means of $\mathcal{F}$ the *semi-inner products* $(\cdot, \cdot)_\pm$ in X are defined as

$$(y, x)_+ = \max\{x^*(y) : x^* \in \mathcal{F}(x)\}, \quad (y, x)_- = \min\{x^*(y) : x^* \in \mathcal{F}(x)\}$$

or, equivalently,

$$(y, x)_+ = |x| \lim_{h \to 0+} \frac{|x + hy| - |x|}{h}, \quad (y, x)_- = |x| \lim_{h \to 0+} \frac{|x| - |x - hy|}{h}.$$

With this notations an operator A in X is accretive iff

$$(y - \overline{y}, x - \overline{x})_+ \geq 0 \quad \text{for all } x,\ \overline{x} \in D(A),\ y \in Ax \text{ and } \overline{y} \in A\overline{x}.$$

Instead of $(\cdot,\cdot)_+$ we will sometimes use the *bracket*, defined by

$$[x,y] = \lim_{h\to 0+} \frac{|x+hy|-|x|}{h};$$

evidently A is accretive iff

$$[x-\overline{x}, y-\overline{y}] \geq 0 \quad \text{for all } x,\ \overline{x} \in D(A),\ y \in Ax \text{ and } \overline{y} \in A\overline{x}.$$

If A has this property then its *resolvents* $J_\lambda := (I+\lambda A)^{-1}$ are single-valued and *nonexpansive* mappings for every $\lambda > 0$, i.e. $J_\lambda : R(I+\lambda A) \to D(A)$ satisfies $|J_\lambda x - J_\lambda \overline{x}| \leq |x-\overline{x}|$. This is clear by the first definition of accretivity above, and in fact the converse also holds. The next result collects some properties of J_λ that are important in the sequel. For simplicity, we only consider the case when A is *m-accretive.* The latter means that A is accretive and satisfies $R(I+\lambda A) = X$ for all (or, equivalently, for some) $\lambda > 0$. We say that A is *m-ω-accretive* (with $\omega \in \mathbb{R}$) if $A+\omega I$ is m-accretive, and then $R(I+\lambda A) = X$ holds for all $\lambda > 0$ such that $\lambda\omega < 1$.

Proposition 1.1 *Let A be m-accretive in a real Banach space X and J_λ denote the resolvents of A. Then the following holds.*
(a) $J_\lambda : X \to D(A)$ is nonexpansive for every $\lambda > 0$.
(b) $|x - J_\lambda x| \leq \lambda |y|$ for all $\lambda > 0$, $x \in D(A)$ and $y \in Ax$.
(c) The J_λ satisfy the resolvent identity, i.e.

$$J_\lambda = J_\mu\left(\frac{\mu}{\lambda} I + \frac{\lambda-\mu}{\lambda} J_\lambda\right) \quad \textit{for all } \lambda, \mu > 0.$$

(d) Let $x_0 \in D(A)$, $y_0 \in Ax_0$ and $\lambda_k > 0$ for $k = 1, \ldots, n$. Then

$$|J_{\lambda_n} J_{\lambda_{n-1}} \cdots J_{\lambda_1} x - x| \leq 2|x-x_0| + \sum_{k=1}^{n} \lambda_k |y_0| \quad \textit{for all } x \in X.$$

The following prototype of a (multivalued) m-accretive operator in $X = \mathbb{R}$ will be important in the sequel.

$$\mathrm{Sgn} : \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset \quad \text{with} \quad \mathrm{Sgn}\,(r) = \begin{cases} 1 & \text{if } r > 0 \\ [-1,1] & \text{if } r = 0 \\ -1 & \text{if } r < 0. \end{cases}$$

In this one-dimensional case an m-accretive operator is often called a *maximal monotone graph in* $\mathbb{R}$. Another example which plays a role in Chapter 3 is given by

$$\beta : \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset \quad \text{with} \quad D(\beta) = \{0\} \text{ and } \beta(0) = \mathbb{R}.$$

Let us also mention that every m-accretive operator A is automatically *maximal accretive*, i.e. A does not admit a proper accretive extension; the converse is (in general) false.

It is sometimes of advantage to approximate an m-accretive operator A by means of the so-called *Yosida approximation* $A_\lambda = \lambda^{-1}(I - J_\lambda)$ with $\lambda > 0$. By means Proposition 1.1 it is obvious that A_λ is single-valued, everywhere defined and Lipschitz of constant $2/\lambda$. It is also not difficult to see that A_λ is m-accretive with $A_\lambda x \in AJ_\lambda x$ on X. Further properties of A_λ can be found in the references mentioned above, but will not be needed here.

If A satisfies the stronger condition

$$(y - \overline{y}, x - \overline{x})_- \geq 0 \quad \text{for all } x,\ \overline{x} \in D(A),\ y \in Ax \text{ and } \overline{y} \in A\overline{x},$$

then A is called s-accretive. This property is in particular important if sums of accretive operators are encountered; notice that $A+B$ is accretive if A is accretive and B is s-accretive. In this context the following result is of interest.

Proposition 1.2 *Let X be a real Banach space and $f : X \to X$ be continuous and accretive. Then f is s-accretive.*

Because of the closed relationship between accretivity and the duality map it is not surprising that certain properties of $\mathcal{F}$ and $(\cdot,\cdot)_\pm$ are important later on, in particular if we consider special Banach spaces with additional smoothness properties. Recall that X is *uniformly convex* if for every $\epsilon \in (0, 2]$ there exists $\delta = \delta(\epsilon) > 0$ such that $|x| = |y| = 1$ and $|x - y| \geq \epsilon$ implies $|(x + y)/2| \leq 1 - \delta$, while X is *strictly convex* if $|x| = |y| = 1$ and $x \neq y$ implies $|\lambda x + (1 - \lambda)y| < 1$ for all $\lambda \in (0, 1)$.

Proposition 1.3 *Let X be a real Banach space and $\mathcal{F} : X \to 2^{X^*} \setminus \emptyset$ the duality map. Then*
(a) $\mathcal{F}(x)$ is convex and $\sigma(X^*, X)$-closed with $\mathcal{F}(\lambda x) = \lambda\mathcal{F}(x)$ for all $\lambda \in \mathbb{R}$.
(b) $\mathcal{F}$ is single-valued iff X^* is strictly convex.
(c) If X^* is strictly convex then $\mathcal{F}$ is continuous from $(X, |\cdot|)$ to $(X^*, \sigma(X^*, X))$.
(d) X^* is uniformly convex iff $\mathcal{F} : X \to X^*$ is uniformly continuous on bounded sets.

For a proof see e.g. Proposition 12.3 and Theorem 12.2 in Deimling [41]. The following properties of $(\cdot,\cdot)_\pm$ are contained in Proposition 13.1 of the same reference.

Proposition 1.4 *Let X be a real Banach space. Then*
(a) $(x, z)_\pm + (y, z)_- \leq (x + y, z)_\pm \leq (x, z)_\pm + (y, z)_+$ and $|(x, y)_\pm| \leq |x|\,|y|$, $(x + \alpha y, y)_\pm = (x, y)_\pm + \alpha|y|^2$ for $\alpha \in \mathbb{R}$, $(\alpha x, \beta y)_\pm = \alpha\beta\,(x, y)_\pm$ for $\alpha\beta \geq 0$.
(b) $(\cdot,\cdot)_+$ is upper semicontinuous, $(\cdot,\cdot)_-$ is lower semicontinuous and $(\cdot, y)_\pm$ is continuous.
(c) If X^* is uniformly convex then $(\cdot,\cdot)_+ = (\cdot,\cdot)_-$ is uniformly continuous on bounded subset of $X \times X$.
(d) If $x : (a, b) \to X$ is differentiable at $t \in (a, b)$ then $\varphi(\cdot) = |x(\cdot)|$ satisfies

$$\varphi(t)D^-\varphi(t) = (x'(t), x(t))_- \quad \text{and} \quad \varphi(t)D^+\varphi(t) = (x'(t), x(t))_+,$$

where

$$D^-\varphi(t) = \overline{\lim_{h\to 0+}}\, \frac{\varphi(t) - \varphi(t - h)}{h} \quad \text{and} \quad D^+\varphi(t) = \overline{\lim_{h\to 0+}}\, \frac{\varphi(t + h) - \varphi(t)}{h}$$

are the Dini-derivatives of φ.

Let us mention some immediate consequences of Proposition 1.3 concerning m-accretive operators. If X^* is uniformly convex and A is m-accretive in X then Ax is closed convex for every $x \in D(A)$ and A is *demiclosed* which means that $(x_n) \subset D(A)$ with $x_n \to x$ and $y_n \in Ax_n$ with $y_n \rightharpoonup y$ implies $x \in D(A)$ and $y \in Ax$. The former property implies that the so-called *minimal section* A^0 of A, defined by

$$A^0 x = \{y \in Ax : |y| = \rho(0, Ax)\},$$

is nonempty on $D(A)$ in this case, and A^0 is single-valued if, in addition, X is strictly convex. Let us also note that $\overline{D(A)}$ is a convex set if A is m-accretive in a real and uniformly convex Banach space.

In many concrete applications the Banach space under consideration is equipped with a natural partial ordering $\leq$. In this situation it is often important to know whether a given operator "respects" this ordering, a question that leads to the concept of T-accretivity. Recall that $(X, |\cdot|, \leq)$ is called a *Banach lattice* if $X^+ = \{x \in X : x \geq 0\}$ is a closed convex cone such that $x \wedge y := \inf\{x, y\}$ and $x \vee y := \sup\{x, y\}$ exist for all $x, y \in X$, and $x \vee (-x) \leq y \vee (-y)$ implies $|x| \leq |y|$. In this case x^+ denotes $x \vee 0$ and x^- is short for $-(x \wedge 0) = (-x)^+$. Given a Banach lattice $(X, |\cdot|, \leq)$, an operator A in X is said to be T-*accretive* if

$$|(x - \overline{x})^+| \leq |(x - \overline{x} + \lambda(y - \overline{y}))^+| \quad \text{for all } \lambda > 0,\ x, \overline{x} \in D(A), y \in Ax, \overline{y} \in A\overline{x}.$$

An equivalent formulation is given by

$$[x - \overline{x}, y - \overline{y}]_+ \geq 0 \quad \text{for all } x, \overline{x} \in D(A), y \in Ax, \overline{y} \in A\overline{x},$$

where $[\cdot, \cdot]_+$ is defined by

$$[x, y]_+ = \lim_{h \to 0+} \frac{|(x + hy)^+| - |x^+|}{h}.$$

If A has this property then the resolvents of A are single-valued and *order-preserving*; the latter means $J_\lambda x \leq J_\lambda \overline{x}$ for all $\lambda > 0$ and $x, \overline{x} \in R(I + \lambda A)$ whenever $x \leq \overline{x}$. Given that A is m-accretive, then A is T-accretive iff the resolvents of A are order-preserving.

In the sequel two special cases will be particularly important, namely $X = \mathbb{R}^m_+$ equipped with the usual componentwise ordering and $X = L^1(\Omega)^m$ where we always consider the partial ordering induced by $X^+ := L^1(\Omega; \mathbb{R}^m_+) = \{u \in X : u_k(x) \geq 0 \text{ a.e. on } \Omega \text{ for } k = 1, \ldots, m\}$. In both situations computation of $[\cdot, \cdot]_+$ is especially simple if the right norms are chosen, for example $|u| = |u_1|_1 + \ldots + |u_m|_1$ in case of $X = L^1(\Omega)^m$. Observe that it is then sufficient to know $[\cdot, \cdot]_+$ for $m = 1$, and in this case we have

$$[a, b]_+ = \max\Big(H(a)\, b\Big) = \begin{cases} b & \text{if } a > 0 \\ b^+ & \text{if } a = 0 \\ 0 & \text{if } a < 0 \end{cases} \qquad \text{if } X = \mathbb{R},$$

where H denotes the Heaviside function with $H(0) = [0,1]$, as well as

$$[u,v]_+ = \max\Big(\int_\Omega H(u)\, v\, dx\Big) = \int_{\{u=0\}} v^+ dx + \int_{\{u>0\}} v dx \quad \text{if } X = L^1(\Omega),$$

where $H(u)$ is short for $\{w \in L^1(\Omega) : w(x) \in H(u(x)) \text{ a.e. on } \Omega\}$.
Let us also note that T-accretivity implies accretivity in the special cases just mentioned. In fact this implication is valid in any Banach lattice $(X, |\cdot|, \leq)$ having the property that $|x^+| \leq |y^+|$ and $|x^-| \leq |y^-|$ implies $|x| \leq |y|$.

1.3 Quasi-autonomous problems

The study of nonlinear perturbations of accretive evolution problems certainly requires considerable knowledge about the simpler *quasi-autonomous problem*

$$u' + Au \ni w(t) \;\text{ on } J = [0,a], \;\; u(0) = u_0 \tag{1}$$

with $w \in L^1(J;X)$ and $u_0 \in \overline{D(A)}$. A function $u : J \to X$ is called a *strong solution* of (1) if u is absolutely continuous with $u(0) = u_0$ and almost everywhere (a.e. for short) differentiable on J such that the inclusion in (1) is satisfied for almost all $t \in J$. Now if A is m-accretive in X one cannot expect (1) to have a strong solution, even if A is linear and densely defined, w is continuous and $u_0 = 0$; see Webb [114] for a counter-example in this setting. It may for example happen that the right candidate u for a solution of (1) is Lipschitz continuous but nowhere differentiable. The latter is not possible if X has the *Radon-Nikodym property* (*RNP* for short), since this property is equivalent to a.e. differentiability of all absolutely continuous functions $u : J \to X$; recall that every reflexive Banach space has the RNP.

For this reason we employ the following concept of a *mild solution* of (1). A continuous function $u : J \to \overline{D(A)}$ with $u(0) = u_0$ is said to be a mild solution of (1) if u is the uniform limit of ϵ-DS-approximate solutions u^ϵ as $\epsilon \to 0+$. Here, by an ϵ-DS-approximate solution u^ϵ of (1) one means a step function u^ϵ with $u^\epsilon(t_0) = u_0$ and $u^\epsilon(t) = u_k$ on $(t_{k-1}, t_k]$ for $k = 1, \ldots, m$, where $0 = t_0 < t_1 < \cdots < t_m < a \leq t_m + \epsilon$ with $t_k - t_{k-1} \leq \epsilon$ and the u_k solve the implicit difference scheme

$$\frac{u_k - u_{k-1}}{t_k - t_{k-1}} + Au_k \ni z_k \quad \text{for } k = 1, \ldots, m$$

with $z_1, \ldots, z_m \in X$ such that $\displaystyle\sum_{k=1}^m \int_{t_{k-1}}^{t_k} |z_k - w(t)|\, dt \leq \epsilon$.

While the concept of mild solutions is of particular importance for theoretical purposes, it is almost impossible to check in practice whether a given $u \in C(J;X)$ is a mild solution. At this point it is useful to introduce still another type of solutions of (1) which is also helpful to obtain uniqueness of mild solutions and for the study of regularity questions. A function

$u \in C(J;X)$ with $u(0) = u_0$ is called an *integral solution* of (1) if the following family of inequalities is satisfied:

$$|u(t)-x| \le |u(s)-x| + \int_s^t [u(\tau)-x, w(\tau)-y]\,d\tau \quad \text{for all } 0 \le s \le t \le a, (x,y) \in A.$$

In the case when A is m-accretive, which will most often be satisfied in the situations to be considered later, the concepts of mild and integral solutions of (1) coincide. The next result collects several facts concerning the quasi-autonomous initial value problem.

Theorem 1.1 *Let A be m-accretive in a real Banach space X, $w \in L^1(J;X)$ with $J = [0,a] \subset \mathbb{R}$ and $u_0 \in \overline{D(A)}$. Then (1) has a unique mild solution u and every sequence of corresponding ϵ_m-DS-approximate solutions u^{ϵ_m} converges to u uniformly on $[0,a)$ if $\epsilon_m \to 0+$. Furthermore u is the unique integral solution of (1). Let u and $\overline{u}$ be mild solutions of (1) for right-hand sides $w, \overline{w} \in L^1(J;X)$ and initial values $u_0, \overline{u}_0$, respectively. Then*

$$|u(t)-\overline{u}(t)| \le |u(s)-\overline{u}(s)| + \int_s^t [u(\tau)-\overline{u}(\tau), w(\tau)-\overline{w}(\tau)]\,d\tau \quad \textit{for all } 0 \le s \le t \le a, \tag{2}$$

in particular

$$|u(t)-\overline{u}(t)| \le |u(s)-\overline{u}(s)| + \int_s^t |w(\tau)-\overline{w}(\tau)|\,d\tau \ \textit{for } 0 \le s \le t \le a.$$

It is not difficult to check that (2) is equivalent to

$$|u(t)-\overline{u}(t)|^2 \le |u(s)-\overline{u}(s)|^2 + 2\int_s^t (w(\tau)-\overline{w}(\tau), u(\tau)-\overline{u}(\tau))_+ d\tau \ \text{ for all } 0 \le s \le t \le a. \tag{3}$$

In the situation of Theorem 1.1 we let $\mathcal{S}w$ denote the mild solution of problem (1), i.e. $\mathcal{S} : L^1(J;X) \to C(J;X)$ is the solution operator of the quasi-autonomous problem associated with A. Then, as a consequence of Theorem 1.1, $\mathcal{S}$ is well-defined and a nonexpansive map. Additional properties of this operator will be obtained in Chapter 2 below, and play a crucial role in the subsequent study of nonlinear evolution problems.

Let us also record the following result on regularity of mild solutions.

Theorem 1.2 *Let A be m-accretive in a real Banach space X and $J = [0,a] \subset \mathbb{R}$. Given $u_0 \in \overline{D(A)}$ and $w \in L^1(J;X)$, let u be the mild solution of (1). Then u is the strong solution of (1) iff u is absolutely continuous and a.e. differentiable on J. In particular, u is the strong solution of (1) if u is Lipschitz continuous and X has the RNP.*

If u is a mild solution of the autonomous initial value problem

$$u' + Au \ni 0 \ \text{ on } \mathbb{R}_+, \ \ u(0) = u_0 \tag{4}$$

with m-accretive A, then u is Lipschitz continuous iff u_0 belongs to the *generalized domain* $\tilde{D}(A) = \{x \in X : |x|_A < \infty\}$ of A and then $|u_0|_A$ is a Lipschitz constant for $u(\cdot)$; here $|\cdot|_A$ is defined by

$$|x|_A = \lim_{r\to 0+} \inf\{|\overline{y}| : \overline{y} \in A\overline{x} \text{ for some } \overline{x} \in B_r(x) \cap D(A)\}.$$

Hence a mild solution u of (4) with $u_0 \in \tilde{D}(A)$ is in fact a strong solution if X has the RNP. More can be said if u_0 belongs to $D(A)$ and X enjoys additional smoothness properties.

Theorem 1.3 *Let X be a real Banach space such that X and X^* are uniformly convex. Let A be m-accretive in X, $u_0 \in D(A)$ and u be the mild solution of (4). Then $u(t) \in D(A)$ on $\mathbb{R}_+$, $u(t)$ has a derivative $u'_+(t)$ from the right at every $t \geq 0$, and $u'_+(\cdot)$ is continuous from the right and satisfies*

$$u'_+(t) + A^0 u(t) = 0 \quad \text{on } \mathbb{R}_+,$$

where $A^0 x$ denotes the unique element of minimal norm of Ax.

Let us note that all results given in the present section remain valid for m-ω-accretive A, except that the inequalities in Theorem 1.1 have to be modified. For example (2) becomes

$$|u(t) - \overline{u}(t)| \leq |u(s) - \overline{u}(s)| + \omega \int_s^t |u(\tau) - \overline{u}(\tau)|\, d\tau + \int_s^t [u(\tau) - \overline{u}(\tau), w(\tau) - \overline{w}(\tau)]\, d\tau$$

for $0 \leq s \leq t \leq a$.

1.4 Nonlinear semigroups

Let X be a real Banach space and $\emptyset \neq D \subset X$. A family $(S(t))_{t\geq 0}$ of functions $S(t) : D \to D$ is called a *semigroup of nonexpansive mappings on D* if

$$S(t+s) = S(t)S(s) \text{ for } t, s \geq 0, \quad S(0) = I, \quad \lim_{t\to 0+} S(t)x = x \text{ on } D$$

and

$$|S(t)x - S(t)\overline{x}| \leq |x - \overline{x}| \text{ for all } t \geq 0 \text{ and } x, \overline{x} \in D.$$

In this case the map $(t, x) \to S(t)x$ is jointly continuous from $\mathbb{R}_+ \times D$ into D.

Now if A is an m-accretive operator in X, then there is a particular semigroup $(S(t))_{t\geq 0}$ on $\overline{D(A)}$ associated with A, which is given by the *exponential formula*

$$S(t)x = \lim_{n\to\infty} J^n_{t/n} x \quad \text{for } t \geq 0 \text{ and } x \in \overline{D(A)},$$

where the convergence is uniform for t from bounded subsets of $\mathbb{R}_+$. In this situation $(S(t))_{t\geq 0}$ is called the semigroup generated by $-A$, and the function $u(t) = S(t)u_0$ with $u_0 \in \overline{D(A)}$ is the mild solution of the autonomous problem (4). Actually, this also holds if A is accretive and satisfies the range condition

$$R(I + \lambda A) \supset \overline{D(A)} \quad \text{for all small } \lambda > 0.$$

In the sequel we simply speak of a semigroup and write $S(t)$ instead of $(S(t))_{t\geq 0}$.

A semigroup $S(t)$ of nonexpansive mappings on $D \subset X$ is said to be *compact* if $S(t)$ is a compact map for every $t > 0$, i.e. if $S(t)B$ is relatively compact in X for all $t > 0$ and bounded $B \subset D$, while $S(t)$ is said to be *equicontinuous* if the family of maps $\{S(\cdot)x : x \in B\}$

is equicontinuous at every $t > 0$ for all bounded $B \subset X$. If A is m-accretive then the semigroup $S(t)$ generated by $-A$ is compact iff $S(t)$ is equicontinuous and A has compact resolvents.

Let us add a few remarks concerning the linear case. Suppose that A is m-accretive, linear and densely defined, and let $S(t)$ be the semigroup generated by $-A$. In this special case $S(t)$ is also called the C_0-*semigroup* generated by $-A$, and the maps $S(t)$ are bounded linear operators on X. Then the semigroup is equicontinuous iff $S(t)$ is continuous in the uniform operator topology for $t > 0$. If $w \in L^1(J; X)$ and u is a strong solution of (1), then it is easy to check that u satisfies the *variation of constants formula*, i.e.

$$u(t) = S(t)u_0 + \int_0^t S(t-s)w(s)\,ds \quad \text{on } J. \tag{5}$$

In the theory of linear evolution equations a mild solution u of (1) is usually defined by means of this formula, and the following result states that this concept of mild solutions coincides with the one defined above.

Proposition 1.5 *Let X be a real Banach space and A be a linear, m-accretive and densely defined operator in X. Then u is a mild solution of (1) iff u satisfies the variation of constants formula (5).*

For more information concerning the linear case we refer to Goldstein [60] and Pazy [93].

Let us finally provide some information concerning perturbations of nonlinear semigroups. For this purpose we need the following notation: if $(A_k)_{k>0}$ is a family of operators in a Banach space X, then $\liminf\limits_{k\to\infty} A_k$ is defined by $(x, y) \in \liminf\limits_{k\to\infty} A_k$ if there are $(x_k, y_k) \in A_k$ for every $k > 0$ such that $x_k \to x$ and $y_k \to y$.

Theorem 1.4 *Let X be real Banach space, $(A_k)_{k\geq 0}$ be a family of ω-accretive operators in X with the same $\omega \in \mathbb{R}$, and $A_\infty \subset \liminf\limits_{k\to\infty} A_k$. Suppose that there exist mild solutions u_k of*

$$u_k'(t) + A_k u_k(t) \ni 0 \quad \text{on } \mathbb{R}_+, \quad u_k(0) = u_0^k$$

for every $k \geq 0$, as well as a mild solution u_∞ of

$$u_\infty'(t) + A_\infty u_\infty(t) \ni 0 \quad \text{on } \mathbb{R}_+, \quad u_\infty(0) = u_0^\infty.$$

Then $u_k \in \overline{D(A_k)}$ with $u_k \to u_\infty \in \overline{D(A_\infty)}$ implies $u_k(t) \to u_\infty(t)$ on $\mathbb{R}_+$, where the convergence is uniform on bounded sets.

Recall that existence of the mild solutions u_k and u_∞ is guaranteed if all A_k as well as A_∞ satisfy the range condition, i.e. if for instance $R(I + \lambda A_\infty) \supset \overline{D(A_\infty)}$ for all $\lambda > 0$ with $\lambda\omega < 1$.

§2 Upper Semicontinuous Differential Inclusions

We continue with the compilation of known facts concerning set-valued operators, but here the emphasis is on continuity properties of such multivalued mappings. As before we concentrate on notations and results that are needed in the sequel; more details and further information can be found in Aubin/Cellina [8] and Deimling [42]. For proofs we also refer to these books, unless an explicit reference is given.

2.1 Upper semicontinuity

Let X, Y be Banach spaces and $\emptyset \neq \Omega \subset Y$. A mapping $F : \Omega \to 2^X$ is called a *multivalued map* (or, simply, a *multi*) with values $F(\omega) \subset X$, and we keep the notations of the domain, range and graph of F. A single-valued $f : \Omega \to X$ with $f(\omega) \in F(\omega)$ on Ω is called a *selection* of F. Here we also need the concept of the "inverse" F^{-1} which is defined by

$$F^{-1}(A) = \{\omega \in \Omega : F(\omega) \cap A \neq \emptyset\} \quad \text{for } A \subset X;$$

A multivalued map $F : \Omega \to 2^X \setminus \emptyset$ is called *upper semicontinuous, usc* for short, if $F^{-1}(A)$ is closed in Ω whenever $A \subset X$ is closed; remember that "Ω_0 closed in Ω" means $\Omega_0 = A \cap \Omega$ for some closed $A \subset Y$. A multi $F : \Omega \to 2^X \setminus \emptyset$ is said to be *ϵ-δ-usc* if for every $\omega_0 \in \Omega$ and $\epsilon > 0$ there exists $\delta = \delta(\omega_0, \epsilon) > 0$ such that $F(\omega) \subset F(\omega_0) + B_\epsilon(0)$ on $B_\delta(\omega_0) \cap \Omega$. Evidently, both concepts reduce to continuity in the single-valued case.
Let us record some elementary properties of usc multis.

Proposition 2.1 *Let X, Y be Banach spaces and $\emptyset \neq \Omega \subset Y$. Then*
(a) If $F : \Omega \to 2^X \setminus \emptyset$ is usc then F is ϵ-δ-usc. The converse is true if F has compact values.
(b) If F is usc with compact values then $F(C)$ is compact for all compact C.
(c) If F is ϵ-δ-usc and Ω is closed then F has closed graph. If $gr(F)$ is closed and $\overline{F(\Omega)}$ is compact then F is usc.
(d) If F is ϵ-δ-usc then $\overline{\text{conv}}F$ is ϵ-δ-usc.

An important example of a multivalued map is the *metric projection* onto a subset C of X, defined by

$$P_C(x) = \{y \in C : |x - y| = \rho(x, C)\}.$$

If $\emptyset \neq C \subset X$ is compact then P_C is usc with compact nonempty values. If C is also convex then $P_C(x)$ is convex. Another example is the set-valued version Sgn of the sign-function mentioned before. The latter is a prototype for multivalued "regularizations" of discontinuous functions $f : \Omega \to X$, obtained by "filling in the gaps at points of discontinuity of f" in simplest cases. More generally, given $F : \Omega \to 2^X \setminus \emptyset$ with closed values, such a regularization $\hat{F}$ of F is obtained by means of

$$\hat{F}(\omega) = \bigcap_{\delta > 0} \overline{\text{conv}} F(B_\delta(\omega) \cap \Omega) \quad \text{for } \omega \in \Omega.$$

If F is locally compact, i.e. if for every $\omega_0 \in \Omega$ there exists $r = r(\omega_0) > 0$ such that $\overline{F(B_r(\omega_0) \cap \Omega)}$ is compact, then $\hat{F}$ is usc with compact convex values $\hat{F}(\omega) \supset F(\omega)$. While upper semicontinuous multis, like P_C or Sgn from above, need not admit continuous selections, the following proposition shows that "approximate selections" exist under additional assumptions.

Proposition 2.2 *Let X, Y be Banach spaces, $\emptyset \neq D \subset X$ compact and $F : D \to 2^X \setminus \emptyset$ usc with compact convex values. Then, given $\epsilon > 0$, there is a continuous $f_\epsilon : D \to X$ such that*

$$f_\epsilon(x) \in F(B_\epsilon(x) \cap D) + B_\epsilon(0).$$

In the situation of Proposition 2.2 with $X = Y$ the mapping F has a fixed point, i.e. there is $x \in D$ such that $x \in F(x)$, if also $F(D) \subset D$ holds. The same is true under weaker assumptions on the values of F.

Lemma 2.1 *Let X be a Banach space, $\emptyset \neq D \subset X$ compact convex and $F : D \to 2^D \setminus \emptyset$ usc with closed contractible values. Then F has a fixed point.*

This is a special case of the Corollary given in Górniewicz/Granas/Kryszewski [62], where the values of F are only assumed to be compact R_δ-sets.

From the viewpoint of applications in infinite dimensional Banach spaces the following weaker version of upper semicontinuity is more appropriate. A multi $F : \Omega \to 2^X \setminus \emptyset$ is said to be *weakly usc* if $F^{-1}(A)$ is closed in Ω for all weakly closed $A \subset X$. Equivalently, F is weakly usc iff $\{x \in \Omega : F(x) \subset V\}$ is open in Ω whenever $V \subset X$ is weakly open. If F has this property then $x^* \circ F : \Omega \to 2^{\mathbf{R}} \setminus \emptyset$ is obviously usc for every $x^* \in X^*$. More information is contained in

Proposition 2.3 *Let X be a Banach space, $\Omega \neq \emptyset$ a subset of another Banach space and $F : \Omega \to 2^X \setminus \emptyset$ have weakly compact values. Then the following holds.*
(a) If F is ϵ-δ-usc then F is weakly usc.
(b) If the values of F are also convex, then F is weakly usc iff $(x_n) \subset \Omega$ with $x_n \to x_0 \in \Omega$ and $y_n \in F(x_n)$ implies $y_{n_k} \rightharpoonup y_0 \in F(x_0)$ for some subsequence (y_{n_k}) of (y_n).

Proof. To obtain part (a), let F be ϵ-δ-usc and suppose that F is not weakly usc, i.e. there is $(x_n) \subset \Omega$ with $x_n \to x_0 \in \Omega$ and a weakly closed $A \subset X$ such that $F(x_n) \cap A \neq \emptyset$ for all $n \geq 1$ and $F(x_0) \cap A = \emptyset$. Let $\epsilon := \inf\{\rho(y, A) : y \in F(x_0)\}$. We are done if $\epsilon > 0$, since then $(F(x_0) + B_\epsilon(0)) \cap A = \emptyset$, hence $F(x_n) \subset F(x_0) + B_\epsilon(0)$ for all large $n \geq 1$ gives the contradiction $F(x_n) \cap A = \emptyset$ for those n. If $\epsilon = 0$, we find $y_n \in F(x_0)$ and $z_n \in A$ such that $|y_n - z_n| \to 0$. Since $F(x_0)$ is weakly compact we may assume $y_n \rightharpoonup y_0 \in F(x_0)$, hence also $z_n \rightharpoonup y_0 \in A$ which gives $y_0 \in F(x_0) \cap A$, a contradiction.

Concerning (b) notice that sufficiency is obvious. To prove necessity let us first show that $F(C)$ is weakly compact for every compact $C \subset \Omega$. For this purpose let $\bigcup_{\lambda \in \Lambda} V_\lambda$ be any weakly open covering of $F(C)$. For any $x \in C$, $F(x)$ is then covered by finitely many V_λ, the union of which we denote by V_x. Since F is weakly usc and V_x is weakly open the sets $U_x := \{\tilde{x} \in \Omega : F(\tilde{x}) \subset V_x\}$ are open in Ω and cover C. Hence $C \subset \bigcup_{i=1}^{m} U_{x_i}$ for certain $x_1, \ldots, x_m \in C$. This yields $F(C) \subset \bigcup_{i=1}^{m} V_{x_i}$ since $y \in F(C)$ means $y \in F(x)$ for some $x \in C$ and $x \in U_{x_i}$ for some i. Therefore, $F(C)$ is weakly compact since $\bigcup_{i=1}^{m} V_{x_i}$ is the union of finitely many V_λ.
Let $(x_n) \subset \Omega$ with $x_n \to x_0 \in \Omega$ and $y_n \in F(x_n)$. Then $F(\overline{\{x_n : n \geq 1\}})$ is weakly compact, hence $y_{n_k} \rightharpoonup y_0$ for some subsequence. Suppose $y_0 \notin F(x_0)$. By Mazur's Theorem we find $x^* \in X^*$ such that $x^*(y) \leq r$ on $F(x_0)$ and $x^*(y_0) \geq r + 2\delta$ with some $r \in \mathbb{R}$ and $\delta > 0$. Then $F(x_{n_k}) \cap A \neq \emptyset$ for all large $k \geq 1$ with the weakly closed set $A = \{y \in X : x^*(y) \geq r + \delta\}$ implies $F(x_0) \cap A \neq \emptyset$, a contradiction. □

Since the concept of *lower semicontinuity* (*lsc* for short) will only play a minor role in the sequel, let us just mention that $F : \Omega \to 2^X \setminus \emptyset$ is called lsc if $F^{-1}(V)$ is open in Ω whenever $V \subset X$ is open. Equivalently, F is lsc iff $\rho(x, F(\cdot))$ is usc for all $x \in X$. Here usc refers to upper semicontinuity of real valued functions; recall that $\varphi : \Omega \to \mathbb{R}$ is usc if $(\omega_n) \subset \Omega$ with $\omega_n \to \omega_0$ implies $\overline{\lim}_{n\to\infty} \varphi(\omega_n) \leq \varphi(\omega_0)$ for every $\omega_0 \in \Omega$. If F is lsc with closed convex values then F admits a continuous selection Michael's selection theorem.

Let us also note that $F : \Omega \to 2^X \setminus \emptyset$ is said to be *continuous (w.r.t. to d_H)* if $(\omega_n) \subset \Omega$ with $\omega_n \to \omega_0$ implies $d_H(F(\omega_n), F(\omega_0)) \to 0$ for every $\omega_0 \in \Omega$.

2.2 Measurability

Let us first introduce some standard notation. The triple $(J, \mathcal{L}, \lambda_1)$ denotes the measure space consisting of the Lebesgue measurable subsets $\mathcal{L}$ of an interval $J \subset \mathbb{R}$ with the one-dimensional Lebesgue measure $\lambda_1 : \mathcal{L} \to \bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. A set $M \in \mathcal{L}$ is simply called measurable in the sequel, and $N \in \mathcal{L}$ is a null set if $\lambda_1(N) = 0$; recall that $(J, \mathcal{L}, \lambda_1)$ is a complete measure space, i.e. subsets of null sets are measurable. The Lebesgue measure on $\mathbb{R}^n$ is denotes by λ_n. Given a Banach space X we let $\mathcal{B}$ be the Borel measurable subsets of X, i.e. $\mathcal{B}$ is the smallest σ-algebra containing all open subsets of X. Then $w : J \to X$ is called measurable if $w^{-1}(B) \in \mathcal{L}$ for all $B \in \mathcal{B}$ which is the same as $w^{-1}(V) \in \mathcal{L}$ for all open $V \subset X$. Moreover, w is strongly measurable if w is measurable and there is a null set $N \subset J$ such that $w(J \setminus N)$ is contained in a separable subspace of X.

Recall that a subset W of $L^1(J; X)$ is called *uniformly integrable* if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $A \in \mathcal{L}$ with $\lambda_1(A) \leq \delta(\epsilon)$ implies $\int_A |w(t)|dt \leq \epsilon$ for all $w \in W$; remember that this is always true if $|w(t)| \leq \varphi(t)$ a.e. on J for all $w \in W$ with some $\varphi \in L^1(J)$.

If $(\Omega, \mathcal{A})$ is a measurable space, a multivalued $F : \Omega \to 2^X \setminus \emptyset$ is called *$\mathcal{A}$-measurable* if $F^{-1}(V) \in \mathcal{A}$ for every open $V \subset X$. In case $\mathcal{A} = \mathcal{L}$, we write "measurable" instead of "$\mathcal{A}$-measurable". If F has closed values and ν is a measure on $\mathcal{A}$, then F is called *strongly measurable* if there is a sequence of step-multis F_n, i.e. multis of type $\sum_{i \geq 1} \chi_{A_i} C_i$ with disjoint $A_i \in \mathcal{A}$ and closed $C_i \in 2^X \setminus \emptyset$, such that $d_H(F_n(\omega), F(\omega)) \to 0$ ν-a.e. on Ω; here χ_A denotes the characteristic function of a set A. Observe that the definition of strong measurability just given is consistent with the corresponding single-valued notion.

In the special case when $F : \Omega \to 2^X \setminus \emptyset$ is defined on an open bounded subset Ω of $\mathbb{R}^n$, it is not difficult to check that strong measurability of F is the same as "F has the *Lusin property*", which means that for every $\epsilon > 0$ there is a closed $\Omega_\epsilon \subset \Omega$ with $\lambda_n(\Omega \setminus \Omega_\epsilon) \leq \epsilon$ such that $F_{|\Omega_\epsilon}$ is continuous (w.r. to d_H). If X is *separable*, i.e. $X = \overline{\{x_k : k \geq 1\}}$ for some sequence $(x_k) \subset X$, the latter is equivalent to measurability of F. In this situation several equivalent characterizations of measurability exist and are recorded below.

Lemma 2.2 *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite complete measure space, X be a separable Banach space and $F : \Omega \to 2^X \setminus \emptyset$ have closed values. Then the following statements are equivalent.*
(a) $F^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$.
(b) $F^{-1}(A) \in \mathcal{A}$ for all closed $A \subset X$.
(c) $F^{-1}(V) \in \mathcal{A}$ for all open $V \subset X$.
(d) $F(\omega) = \overline{\{f_n(\omega) : n \geq 1\}}$ on Ω with a sequence of measurable selections f_n of F.
(e) $\rho(x, F(\cdot))$ is measurable for all $x \in X$.
(f) $\mathrm{graph}(F) \in \mathcal{A} \otimes \mathcal{B}$, *where $\mathcal{A} \otimes \mathcal{B}$ is the smallest σ-algebra containing all $A \times B$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.*

This is Theorem III.30 in Castaing/Valadier [34]. Observe that Lemma 2.2 in particular says that a measurable F with closed values in a separable Banach space admits a (strongly) measurable selection. A second consequence of this result will also be important later on: if, under the assumptions of Lemma 2.2, F and G are $\mathcal{A}$-measurable such that $F(\omega) \cap G(\omega) \neq \emptyset$ μ-a.e. on Ω, then $F \cap G$ is $\mathcal{A}$-measurable, too.

2.3 Existence of viable solutions

A large part of the subsequent chapter is devoted to the study of accretive evolution problems with multivalued perturbations of usc type, for example initial value problems of the form

$$u' \in -Au + F(t, u) \ \text{ on } J = [0, a], \ \ u(0) = u_0$$

with m-accretive A and a multi F. In addition, we are often interested in solutions that satisfy additional constraints, say $u(t) \in K$ on J with a closed subset K of X in the simplest case. In the present section we introduce some further concepts that are important for the study of such kind of problems. To motivate these notations we consider the special case $A = 0$, i.e.

$$u' \in F(t,u) \ \text{ on } J, \ \ u(0) = u_0. \tag{1}$$

In analogy to the case of evolution equations governed by m-accretive operators we say that $u : J \to X$ is a *strong solution* of (1) if u is absolutely continuous and a.e. differentiable such that the inclusion in (1) is satisfied for almost all $t \in J$; notice that such a differential inclusion usually does not admit continuously differentiable solutions. Let us note that strong solutions are also called absolutely continuous solutions in this setting.

If F is only defined on $J \times K$ with a closed subset $K \subset X$, then every solution u of (1) also has to satisfy $u(t) \in K$ on J which evidently requires certain conditions at the boundary of K: let $t \in [0,a)$ be such that $y := u'(t)$ exists and satisfies $y \in F(t,x)$ with $x := u(t)$. Then

$$u(t+h) = x + hy + o(h) \in K \ \text{ for all small } h > 0$$

with $o(h)/h \to 0$ as $h \to 0+$. Hence y is subtangential to K at x, i.e. $y \in T_K(x)$ with the tangent cone $T_K(x)$ defined by

$$T_K(x) = \{y \in X : \underline{\lim}_{h \to 0+} h^{-1}\rho(x+hy, K) = 0\} \ \text{ for } x \in K. \tag{2}$$

Let us mention that $T_K(x)$ is sometimes called the Bouligand contingent cone. Consequently, an almost necessary condition for existence of strong solutions of (1) is given by

$$F(t,x) \cap T_K(x) \neq \emptyset \ \text{ on } [0,a) \times K;$$

in the single-valued case the latter reduces to a condition of "Nagumo type".

Let us summarize some basic properties of this tangent cone.

Proposition 2.4 *Let X be a real Banach space and $\emptyset \neq K \subset X$. Then*

(a) $0 \in T_K(x)$ on K and $T_K(x) = X$ on $\overset{\circ}{K}$. The sets $T_K(x)$ are closed with $\lambda T_K(x) \subset T_K(x)$ for all $\lambda \geq 0$.

(b) If K is convex then $T_K(x)$ is convex and given as

$$T_K(x) = \{y \in X : \lim_{h \to 0+} h^{-1}\rho(x+hy, K) = 0\} \ \text{ on } K.$$

(c) If K is convex then $T_K(x) = \overline{\{\lambda(y-x) : \lambda \geq 0, y \in K\}}$ and $T_K(\cdot)$ is lsc on K.

In the sequel we will also consider time-dependent constraints of type $u(t) \in K(t)$ where $K : J \to 2^X \setminus \emptyset$ is a given "tube" such that $\mathrm{gr}\,(K)$ is *closed from the left*, i.e. $t_n \nearrow t$ and

$x_n \in K(t_n)$ with $x_n \to x$ implies $x \in K(t)$. By the considerations given above it is clear that $T_K(x)$ then has to be replaced by the set

$$T_K(t,x) = \{y \in X : \varliminf_{h \to 0+} h^{-1}\rho(x+hy, K(t+h)) = 0\} \text{ for } t \in [0,a) \text{ and } x \in K,$$

in order to formulate the corresponding subtangential condition.

Additional assumptions on F have to be imposed if initial value problem (1) is considered in an infinite dimensional Banach space. Recall that even the single-valued autonomous problem

$$u' = f(u) \text{ on } J, \ \ u(0) = u_0 \tag{3}$$

need not have a (local) solution for continuous $f : X \to X$, but local existence is guaranteed if, e.g., f is locally Lipschitz of compact. A combination of compactness and Lipschitz conditions is possible by means of *measures of noncompactness*. Let X be an infinite dimensional Banach space and $\mathcal{B}$ be the family of bounded subsets of X. Then $\alpha : \mathcal{B} \to \mathbb{R}_+$, defined by

$$\alpha(B) = \inf\{d > 0 : B \text{ admits a finite cover by sets of diameter } \le d\} \text{ for } B \in \mathcal{B},$$

is the *Kuratowski-measure of noncompactness*, and $\beta : \mathcal{B} \to \mathbb{R}_+$, defined by

$$\beta(B) = \inf\{r > 0 : B \text{ can be covered by finitely many balls of radius } \le r\} \text{ for } B \in \mathcal{B},$$

is called *Hausdorff-measure of noncompactness*. These functions have the following properties.

Proposition 2.5 *Let X be a Banach space of infinite dimension, $\mathcal{B}$ the family of bounded subsets of X, and $\gamma : \mathcal{B} \to \mathbb{R}_+$ be either $\alpha(\cdot)$ or $\beta(\cdot)$. Then*
(a) $\gamma(B) = 0$ iff $\bar{B}$ is compact.
(b) γ is a seminorm, i.e. $\gamma(\lambda B) = |\lambda|\gamma(B)$ and $\gamma(B_1 + B_2) \le \gamma(B_1) + \gamma(B_2)$.
(c) $B_1 \subset B_2$ implies $\gamma(B_1) \le \gamma(B_2)$; $\gamma(B_1 \cup B_2) = \max\{\gamma(B_1), \gamma(B_2)\}$.
(d) $\gamma(\mathrm{conv} B) = \gamma(B)$, $\gamma(\bar{B}) = \gamma(B)$.
(e) γ is continuous with respect to d_H.
(f) $\alpha(B_r(x)) = 2r$ and $\beta(B_r(x)) = r$.

In general one has $\beta(B) \le \alpha(B) \le 2\beta(B)$ and both inequalities may be strict. Notice also that $\beta(B)$ depends on the sets from which the centers of the covering balls are chosen. Therefore, if $B \subset \Omega \subset X$ and centers are chosen from Ω instead of X, we write $\beta_\Omega(B)$ and have $\beta(B) \le \beta_\Omega(B) \le 2\beta(B)$ for all bounded $B \subset \Omega$.

Let us note in passing that $F = F_1 + F_2$ with $F_1, F_2 : K \subset X \to 2^X \setminus \emptyset$ satisfies $\beta(F(B)) \le k\beta(B)$ for bounded $B \subset K$, if $\overline{F_1(B)}$ is compact for bounded $B \subset K$ and F_2 has relatively compact values such that $d_H(F_2(x), F_2(y)) \le k|x-y|$ for all $x, y \in K$.

The subsequent result establishes existence of strong solutions of (1) in a quite general situation. In Theorem 2.1 below a multi $F : G \to 2^X \setminus \emptyset$, defined on the graph G of a tube $K : J \to 2^X \setminus \emptyset$, is said to be *almost usc* if for every $\epsilon > 0$ there is a closed $J_\epsilon \subset J$ with $\lambda_1(J \setminus J_\epsilon) \le \epsilon$ such that $F_{|[J_\epsilon \times X] \cap G}$ is usc.

Theorem 2.1 *Let X be a Banach space, $J = [0,a] \subset \mathbb{R}$ and $K : J \to 2^X \setminus \emptyset$ be such that $gr(K)$ is closed from the left. Let $F : gr(K) \to 2^X \setminus \emptyset$ be almost usc with closed convex values such that*

$$||F(t,x)|| \le c(t)(1 + |x|) \quad \text{on } gr(K) \text{ with } c \in L^1(J)$$

and

$$\lim_{h \to 0+} \beta(F([J_{t,h} \times B] \cap gr(K)) \le k(t)\beta(B) \quad \text{on } (0,a] \text{ for bounded } B \subset X$$

with $k \in L^1(J)$, where $J_{t,h} = [t-h,t] \cap J$. Suppose that

$$F(t,x) \cap T_K(t,x) \neq \emptyset \quad \text{for all } t \in [0,a) \setminus N, x \in K(t)$$

and $T_K(t,x) \neq \emptyset$ for all $t \in N$ and $x \in K(t)$ with some null set $N \subset [0,a)$.
Then (1) has a strong solution on J for every $u_0 \in K(0)$.

This is Theorem 1 in Bothe [19]. For later use we finally record the following consequence of Theorem 2.1; recall that $D \subset X$ is locally closed if $D \cap \overline{B}_r(x)$ is closed for every $x \in D$ with some $r = r(x) > 0$.

Corollary 2.1 *Let $D \subset \mathbb{R}^n$ be locally closed and $f : D \to \mathbb{R}^n$ be locally Lipschitz continuous with $f(x) \in T_D(x)$ on D. Then (3) has a unique local solution for every $u_0 \in D$.*

Concerning the proof of Corollary 2.1 let us just note that Theorem 2.1 applies to F, $K(\cdot)$ if we let $K(t) \equiv K := D \cap \overline{B}_r(u_0)$ with sufficiently small $r > 0$ and $F(x) = \varphi(|x - u_0|)f(x)$ where φ is Lipschitz continuous with $\varphi(s) = 1$ on $[0, r/2]$ and $\varphi(s) = 0$ for $s \ge r$; observe that $F(x) \in T_K(x)$ on K by Proposition 2.4. Hence (3) with f replaced by F has a solution u on J and u is obviously a solution of the original problem on $[0,b]$ with some $b > 0$. This solution is unique since F is locally Lipschitz.

Chapter 2

Existence and Qualitative Theory

We consider abstract evolution problems of the type

$$u' + Au \ni f(t,u) \quad \text{on } J = [0,a], \;\; u(0) = u_0, \tag{2.1}$$

where A is m-accretive in a real Banach space X and f is a nonlinear perturbation defined on a certain subset G of $J \times X$. As mentioned before, it may be favorable to replace a discontinuous f by a multivalued "regularization" $F : G \to 2^X \setminus \emptyset$ having some kind of upper semicontinuity property. We also study this case and, since A can be a multivalued operator as well, we then write

$$u' \in -Au + F(t,u) \quad \text{on } J, \;\; u(0) = u_0. \tag{2.2}$$

In the present chapter we start with existence theory in case that the perturbation is defined on all of $J \times \overline{D(A)}$. In this situation existence of a (unique) local mild solution of (2.1) is well known if f is locally Lipschitz continuous, while a local solution need not exist if f is merely continuous and $\dim X = \infty$ even in the special case $A = 0$ (see Godunov [61]). To prove existence of mild solutions we mainly impose additional compactness assumptions, either on the semigroup generated by $-A$ or on the perturbation, where we concentrate on initial value problem (2.2) with F being of upper semicontinuous type. The reason to start directly with the multivalued case is twofold. Firstly, problem (2.1) with continuous f is a special case, hence the subsequent results concerning (2.2) carry over immediately, or (under weaker assumptions) by simple modifications of the proofs. This way we will recover the known "single-valued" results, sometimes even generalizations thereof. Secondly, the interplay between accretivity and (upper) semicontinuity is more delicate and leads to interesting additional difficulties. Actually, problem (2.2) need not admit a mild solution even in finite dimensions. However, existence of mild solutions can be established if X has certain smoothness properties or A, F satisfy additional assumptions that are fulfilled in several types of applications.

Another question enters the picture naturally if, due to the physical background, solutions are only meaningful if additional constraints are satisfied. For instance if (2.1) is the abstract formulation of a reaction-diffusion system then u usually represents certain concentrations, hence nonnegativity is a minimal requirement. A further common feature in such kind of applications is that the perturbation, which refers to the reaction part, is only defined on certain thin subsets. Therefore one is often interested in mild solutions that satisfy $u(t) \in K$ for a certain subset K of X. Such questions of invariance or viability are studied in §4, with main emphasis on the single-valued case. If f is continuous (or even locally Lipschitz), we can allow for time-dependent constraints of type $u(t) \in K(t)$, which are incorporated into the evolution problem by assuming that f is only defined on the set $\mathrm{gr}\,(K) \subset J \times X$. Of course a certain "subtangential condition" is then required to have existence of mild solutions. We show that this necessary condition is in fact also sufficient for existence of solutions in several situations, where a basic step consists in obtaining appropriate approximate solutions. If f is only Carathéodory, corresponding results are provided for fixed constraints $K(t) \equiv K$.

Besides the case when f is locally Lipschitz or satisfies a condition of dissipative type, we again impose additional compactness assumptions on A or on f if constraints are present. As a consequence of these conditions, the set $\mathcal{U}(u_0)$ of all mild solutions of (2.1), respectively (2.2) is a compact subset of $C(J;X)$ in all the situations that are considered in §3 and §4. In the first section of §5 it will be shown that $\mathcal{U}(u_0)$ is in fact a compact R_δ-set (especially connected) within the settings of §3. If constraints are present the situation becomes more difficult, but the same conclusion is valid if, in particular, $K(t) \equiv K$ is closed convex and the following "separated subtangential condition" holds: $(I + \lambda A)^{-1} K \subset K$ and $F(t,x) \cap T_K(x) \neq \emptyset$ on $[0,T) \times K$. The proof is based on a recent result concerning locally Lipschitz approximate selections of F, which is also useful for studying the problem of existence of T-periodic solutions in K. The latter is done in §5.2 for T-periodic and Carathéodory f, where sufficient conditions for existence of a T-periodic solution are obtained within two different settings: X is any Banach space and the separated subtangential condition from above is satisfied; X and X^* are uniformly convex, K has nonempty interior and $f(t,x) - Ax \subset T_K(x)$ for $t \in [0,T)$ and $x \in K \cap D(A)$.

The starting point of the final section is the well-known fact that $A + F$ is m-accretive, given that A is m-accretive and $F : X \to X$ is continuous and accretive. This result is extended to usc $F : X \to 2^X \setminus \emptyset$ with compact convex values in §5.3. While this question is not directly related to the previous topics, such criteria for m-accretivity of the sum of two operators can of course be helpful to check whether a concrete application admits an abstract formulation of the type (2.1) or (2.2).

§3 Existence of Solutions

Let A be m-accretive in a real Banach space X, $F : J \times \overline{D(A)} \to 2^X \setminus \emptyset$ with $J = [0,T] \subset \mathbb{R}$ and consider the initial value problem

$$u' \in -Au + F(t,u) \quad \text{on } J, \;\; u(0) = u_0. \tag{1}$$

Given $u_0 \in \overline{D(A)}$, we look for mild solutions of (1) which means that $u \in C(J;X)$ is the mild solution of the quasi-autonomous problem

$$u' + Au \ni w(t) \quad \text{on } J, \;\; u(0) = u_0 \tag{2}$$

with some

$$w \in \operatorname{Sel}(u) := \{v \in L^1(J;X) : v(t) \in F(t,u(t)) \text{ a.e. on } J\}.$$

Evidently u is a mild solution of (1) iff $u \in C(J;X)$ is a fixed point of $G := \mathcal{S} \circ \operatorname{Sel}$, where $\mathcal{S}w$ denotes the unique mild solution of (2) corresponding to $w \in L^1(J;X)$ for fixed $u_0 \in \overline{D(A)}$. To obtain existence of mild solutions by means of this fixed point approach, additional assumptions are of course needed even for single-valued continuous perturbations. In the multivalued case a further difficulty occurs, since the graph of G need not be closed. Notice that, even if F is usc with compact convex values, $(w_n) \subset \operatorname{Sel}(u_n)$ with $u_n \to u$ in $C(J;X)$ only implies weak relative compactness of (w_n) in $L^1(J;X)$. Then the crucial point is whether $w_n \rightharpoonup w$ in $L^1(J;X)$ and $\mathcal{S}w_n \to u$ in $C(J;X)$ implies $\mathcal{S}w = u$. This does not hold in general, and the following counter-example shows that this difficulty is not a purely proof-technical one.

3.1 Nonexistence.

Even in finite dimensions, initial value problem (1) need not have a mild solution. More precisely, within the next example we define an m-accretive operator A in $\mathbb{R}^4$ and a bounded usc F with compact convex values such that (1) has no mild solution.

Example 3.1 (a) We start with a two-dimensional example in which $w_n \rightharpoonup w$ in $L^1(J;X)$ and $\mathcal{S}w_n \to u$ in $C(J;X)$ do not imply $\mathcal{S}w = u$.
Let $X = \mathbb{R}^2$ with norm $|x|_1 = |x_1| + |x_2|$, $z = (1,0)$ and $A : X \to 2^X \setminus \emptyset$ be given by

$$Ax = \begin{cases} \{-z\} & \text{if} \quad x_1 < 0 \\ \{(s,\varphi(s)) : -1 \le s \le 1\} & \text{if} \quad x_1 = 0, \\ \{z\} & \text{if} \quad x_1 > 0 \end{cases}$$

where $\varphi : [-1,1] \to \mathbb{R}$ is Lipschitz of constant 1 with $\varphi(-1) = \varphi(1) = 0$. To show that A is m-accretive in X, let $x, \overline{x} \in X$ and $y \in Ax$, $\overline{y} \in A\overline{x}$ where it suffices to consider the case $\overline{x}_1 \le 0 \le x_1$. By definition of A it is easy to check that

$$[x - \overline{x}, y - \overline{y}] = \max\Big((s - \overline{s}) \operatorname{Sgn}(x_1 - \overline{x}_1)\Big) + \max\Big((\varphi(s) - \varphi(\overline{s})) \operatorname{Sgn}(x_2 - \overline{x}_2)\Big),$$

where $|s|, |\overline{s}| \leq 1$ and $s = 1$ if $x_1 > 0$ as well as $\overline{s} = -1$ if $\overline{x}_1 < 0$. In case $\overline{x}_1 = x_1 = 0$ this yields

$$[x - \overline{x}, y - \overline{y}] \geq |s - \overline{s}| - |\varphi(s) - \varphi(\overline{s})| \geq 0.$$

In the remaining cases it follows that

$$[x - \overline{x}, y - \overline{y}] \geq s - \overline{s} - |\varphi(s) - \varphi(\overline{s})| \geq 0,$$

since then $\overline{s} \leq s$. Consequently, A is accretive and it remains to show that $R(I + A) = X$. Given $y \in X$ with $|y_1| > 1$ it is obvious that $x = y \pm z$ if $\pm y_1 < -1$ is a solution of $x + Ax \ni y$. In case $|y_1| \leq 1$ consider $x = (0, y_2 - \varphi(y_1))$. Then $y - x = (y_1, \varphi(y_1)) \in Ax$.

Let $J = [0, 1]$, $r_n(t) = \operatorname{sgn}(\sin(2^n \pi t))$ be the Rademacher functions (with $\operatorname{sgn}(0) = 1$) and define $(w_n) \subset L^1(J; X)$ by $w_n(t) = r_n(t) z$ on J. Then $w_n(t) \in \{-z, z\}$ on J and $w_n \rightharpoonup 0$ in $L^1(J; X)$ as $n \to \infty$. Due to $\pm z \in A(0)$, the initial value problems

$$u' + Au \ni w_n(t) \ \text{ on } J, \quad u(0) = 0$$

have the strong solution $u = 0$. Hence $\mathcal{S} w_n = 0$ for all $n \geq 1$, since strong solutions are also mild solutions. Therefore $w_n \rightharpoonup 0$ in $L^1(J; X)$ and $\mathcal{S} w_n \to 0$ in $C(J; X)$.
Nevertheless, $\mathcal{S} 0 \neq 0$ unless $\varphi(0) = 0$. Notice that the solution of $u' + Au \ni 0$ on J, $u(0) = 0$ is given by $u(t) = (0, t\,\varphi(0))$ on J.

(b) We will now define an m-accretive operator A and a compact convex $C \subset X$ such that $\mathcal{S}(W)$ is not closed, where $W = \{w \in L^1(J; X) : w(t) \in C \text{ a.e. on } J\}$.
For this purpose let $X = \mathbb{R}^4$, equipped with the norm $|\cdot|_1$, and define $A : X \to 2^X \setminus \emptyset$ by means of $\operatorname{gr}(A) = \operatorname{gr}(A_1) \times \operatorname{gr}(A_2)$, where the A_k correspond to functions φ_k and are given as in part (a). As φ_1 and φ_2 we choose (see Figure 1)

$$\varphi_1(r) = 1 - |r| \ \text{ on } [-1, 1] \quad \text{and} \quad \varphi_2(r) = \begin{cases} 1 + r & \text{if } \ -1 \leq r < \frac{1}{2} \\ -r & \text{if } \ |r| \leq \frac{1}{2} \\ -1 + r & \text{if } \ \frac{1}{2} < r \leq 1. \end{cases}$$

Given $e = (\frac{1}{2}, 0, \frac{1}{2}, 0)$, we let $w_n(t) = r_n(t) e$ on $J = [0, 1]$ with the r_n from above. For $n \geq 1$, let $u_n = \mathcal{S} w_n$ with initial value $u_n(0) = 0$. Since all u_n are Lipschitz continuous mild solutions and X has the Radon-Nikodym property, it follows that the u_n are strong solutions; see Theorem 1.2. By definition of A_k outside the subspace $X_0 = \{x \in X : x_1 = x_3 = 0\}$ and the choice of e it follows that the solutions remain in X_0. Notice, for example, that $u'(t) + Au(t) \ni w(t)$ a.e. on J implies

$$\frac{d}{dt}|u_1(t)| = u_1'(t) \operatorname{sgn} u_1(t) \leq 0 \ \text{ a.e. on } J \text{ whenever } |w_1(t)| \leq 1 \text{ a.e. on } J.$$

Therefore

$$-u_n'(t) \in [Au_n(t) - w_n(t)] \cap X_0 = [\operatorname{gr}(\varphi_1) \times \operatorname{gr}(\varphi_2) - r_n(t) e] \cap X_0 \ \text{ a.e. on } J,$$

hence $u_n'(t) = (0, -\frac{1}{2}, 0, \frac{1}{2} r_n(t))$ a.e. on J. Consequently, $u_n(t) = -ty + (0,0,0,v_n(t))$ on J with

$$y = \left(0, \frac{1}{2}, 0, 0\right) \quad \text{and} \quad v_n(t) = \frac{1}{2}\int_0^t r_n(\tau)d\tau.$$

This shows that $u_n \to u$ in $C(J;X)$ where $u(t) = -ty$ on J. Moreover, $(u_n) \subset \mathcal{S}(W)$ if we let $C = \text{conv}\{-e, e\}$ and $W = \{w \in L^1(J;X) : w(t) \in C \text{ a.e. on } J\}$. Now suppose that $u = \mathcal{S}w$ for some $w \in W$. Then

$$u'(t) + Au(t) \ni w(t) \ \text{ a.e. on } J, \ \ u(0) = 0,$$

which means $y + r(t)e \in Au(t)$ a.e. on J with a measurable $r : J \to [-1,1]$. By definition of A this yields

$$y + \mu e \in \text{gr}\,(\varphi_1) \times \text{gr}\,(\varphi_2) \ \text{ for some } \mu \in [-1,1],$$

hence the contradiction $\varphi_1(s) = \frac{1}{2}$ and $\varphi_2(s) = 0$ with the same $s = \frac{\mu}{2}$.

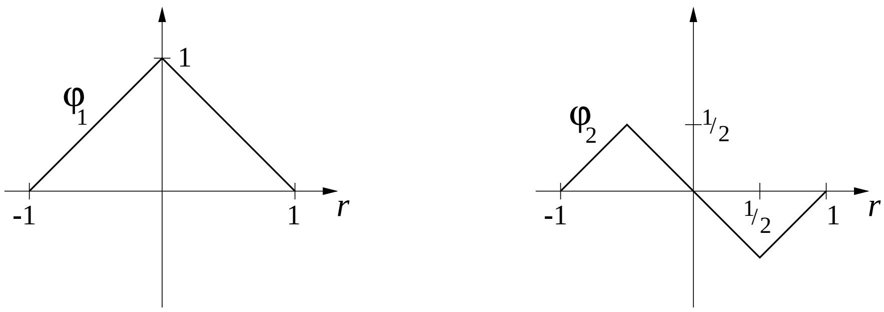

Figure 3.1

(c) Based on the previous part we obtain the following example for nonexistence. Let A, y, e and C be as in (b) and define $F : J \times X \to 2^X \setminus \emptyset$ by

$$F(t,x) = \begin{cases} R(\alpha_k t)e & \text{if } \frac{1}{k+1} < |x + ty| < \frac{1}{k} \\ C & \text{if } |x + ty| = 0, \end{cases}$$

where $\alpha_k = k(k+1)$ and $R(s) = \text{Sgn}\,(\sin(\pi s))$ with $\text{Sgn}\,(\rho) = \rho/|\rho|$ for $\rho \neq 0$ and $\text{Sgn}\,(0) = [-1,1]$. Evidently F is usc and bounded with compact convex values.
Assume that (1) with $u_0 = 0$ has a mild solution u on $J = [0,1]$. As in (b) it follows that $u(\cdot) \in X_0$, and $u(t) = -ty$ is not possible due to $F(t, -ty) \equiv C$.
Let $\psi(t) = |u(t) + ty|$ on J. Due to continuity of ψ, $\psi(0) = 0$ and $\psi \neq 0$ there exist $\tau \in (0,1]$ and $k \geq 1$ such that $\psi(\tau) = \frac{1}{k}$ and $\psi(t) < \frac{1}{k}$ on $[0,\tau)$. Let $\sigma = \max\{t \in [0,\tau] : \psi(t) \leq \frac{1}{k+1}\}$.
Then

$$\psi(\sigma) = \frac{1}{k+1}, \ \ \psi(\tau) = \frac{1}{k} \ \text{ and } \ \frac{1}{k+1} < \psi(t) < \frac{1}{k} \ \text{ on } (\sigma,\tau).$$

Consequently $F(t, u(t)) = R(\alpha_k t)e$ on (σ,τ), which implies

$$u' + Au \ni r_0(\alpha_k t)e \ \text{ a.e. on } [\sigma,\tau].$$

As in the previous step this yields $u'(t) = (0, -\frac{1}{2}, 0, \frac{1}{2} r_0(\alpha_k t))$ a.e. on $[\sigma, \tau]$, hence

$$u(\tau) + \tau y = u(\sigma) + \sigma y + \Big(0, 0, 0, \frac{1}{2} \int_\sigma^\tau r_0(\alpha_k t) dt\Big)$$

and therefore the contradiction

$$\frac{1}{k} = \psi(\tau) \le \psi(\sigma) + \frac{1}{2}\Big| \int_\sigma^\tau r_0(\alpha_k t) dt\Big| \le \frac{1}{k+1} + \frac{1}{2\alpha_k} = \frac{1}{k+1}(1 + \frac{1}{2k}) < \frac{1}{k}.$$

Thus (1) has no mild solution for this choice of A, F and u_0. $\diamond$

This counter-example is essentially Example 1 in Bothe [23], but with appropriate modifications to simplify in particular the verification of m-accretivity. The starting point, i.e. the type of operator considered in step (a), is based on an example in Crandall/Liggett [38], where it was shown by means of similar m-accretive operators in $(\mathbb{R}^2, |\cdot|_\infty)$ that a nonlinear semigroup need not have a unique generator.

3.2 Perturbations of compact semigroups.

We consider initial value problem (1) in the situation when $-A$ generates a compact semigroup and, to avoid problems concerning the continuation of local solutions, we impose the growth condition

$$\|F(t,x)\| := \sup\{|y| : y \in F(t,x)\} \le c(t)(1 + |x|) \quad \text{on } D(F) \text{ with } c \in L^1(J). \tag{3}$$

Once the global results are proved corresponding local versions follow easily as explained in Remark 3.2. In this case the subsequent compactness result immediately yields a compact convex subset of $C(J;X)$ which is invariant under $G = \mathcal{S} \circ \mathrm{Sel}$.

Lemma 3.1 *Let A be m-accretive in a real Banach space X such that $-A$ generates a compact semigroup and let $W \subset L^1(J;X)$ be uniformly integrable. Then $\mathcal{S}(W)$ is relatively compact in $C(J;X)$.*

This is Theorem 2.3.3 in Vrabie [112] which is based on Theorem 2 in Baras [12], where W is of the special type $W = \{w \in L^1(J;X) : |w(t)| \le \varphi(t) \text{ a.e. on } J\}$ with $\varphi \in L^1(J)$.

In the sequel, it will always be assumed that the values of F are at least weakly relatively compact. Then the following criterion for weak relative compactness in $L^1(J;X)$ applies to $\mathrm{Sel}\,(u)$.

Lemma 3.2 *Let X be a Banach space, $J = [0,a] \subset \mathbb{R}$ and $W \subset L^1(J;X)$ be uniformly integrable. Suppose that there exist weakly relatively compact sets $C(t) \subset X$ such that $w(t) \in C(t)$ a.e. on J, for all $w \in W$. Then W is weakly relatively compact in $L^1(J;X)$.*

This is Corollary 2.6 in Diestel/Ruess/Schachermayer [48] specialized to Lebesgue measure.

Existence of a mild solution of problem (1) can be established by means of the fixed point approach in several situations. To avoid repetition of the same arguments under different assumptions, we start with a basic existence result in which closedness of $\mathrm{gr}\,(G)$ is assumed. Afterwards, several sufficient conditions will be given which imply that G has this property.

Lemma 3.3 *Let X be a real Banach space and A be m-accretive such that $-A$ generates a compact semigroup. Let $J = [0,a] \subset \mathbb{R}$ and $F : J \times \overline{D(A)} \to 2^X \setminus \emptyset$ with weakly compact convex values be such that $F(\cdot,x)$ has a strongly measurable selection for every $x \in \overline{D(A)}$, $F(t,\cdot)$ is weakly usc for almost all $t \in J$ and (3) is satisfied. In addition, suppose that $gr\,(G)$ is closed. Then (1) has a mild solution for every $u_0 \in \overline{D(A)}$.*

Proof. 1. We first show that F admits an extension $\tilde{F} : J \times X \to 2^X \setminus \emptyset$ having the same properties as F, such that the solution set of (1) remains the same if F is replaced by $\tilde{F}$. This can be achieved by means of $\tilde{F}(t,x) = F(t,Px)$ on $J \times X$ with $P : X \to \overline{D(A)}$ given by $Px = J_{\lambda(x)}x$ with $J_\lambda = (I + \lambda A)^{-1}$, where we let $\lambda(x) = \rho(x, \overline{D(A)})$ on X and $J_0 x := x$ on $\overline{D(A)}$.
Evidently $\tilde{F}$ has the same properties as F if P is continuous with $|Px| \le c_1 + c_2|x|$ on X, for some c_1, $c_2 > 0$. To prove continuity of P, let $(x_n) \subset X$ with $x_n \to x_0$, $\lambda_n := \lambda(x_n)$ and $\lambda_0 := \lambda(x_0)$. Then $\lambda_n \to \lambda_0$, and $\lambda_0 = 0$ implies $x_0 \in \overline{D(A)}$, hence

$$|Px_n - Px_0| = |J_{\lambda_n}x_n - x_0| \le |x_n - x_0| + |J_{\lambda_n}x_0 - x_0| \to 0$$

in this case. If $\lambda_0 > 0$ then $\lambda_n > 0$ for all large n, hence

$$|Px_n - Px_0| \le |x_n - x_0| + |J_{\lambda_n}x_0 - J_{\lambda_0}x_0| \le |x_n - x_0| + |1 - \frac{\lambda_n}{\lambda_0}|(|x_0| + |J_{\lambda_0}x_0|) \to 0,$$

where the last inequality follows from the resolvent identity. To obtain the estimate for P, fix $\hat{x} \in D(A)$ and $\hat{y} \in A\hat{x}$. Then

$$|Px| \le |x - \hat{x}| + |J_{\lambda(x)}\hat{x}| \le |x - \hat{x}| + \lambda(x)|\hat{y}| + |\hat{x}| \le c_1 + c_2|x| \quad \text{on } X,$$

where $c_1 := |\hat{x}|(2 + |\hat{y}|)$ and $c_2 := 1 + |\hat{y}|$.
Therefore, in the subsequent steps, we may assume that F is defined on $J \times X$; notice that every mild solution u of (1) with $\tilde{F}$ instead of F satisfies $u(J) \subset \overline{D(A)}$, hence u is in fact a solution of the original problem. Moreover, the graph of $\mathcal{S} \circ \mathrm{Sel}$ is also closed if Sel corresponds to $\tilde{F}$ instead of F.

2. We may assume that $F(t,\cdot)$ is usc for all $t \in J$, since a change of $F(t,\cdot)$ for t from a null set does not affect the solutions of (1). To obtain a fixed point of $G = \mathcal{S} \circ \mathrm{Sel}$, let us first show $\mathrm{Sel}\,(u) \ne \emptyset$ for every $u \in C(J;X)$. For this purpose let $u \in C(J;X)$, u_n be step-functions with $|u - u_n|_0 \to 0$ and w_n be strongly measurable selections of $F(\cdot, u_n(\cdot))$.

Then $\{w_n : n \geq 1\} \subset L^1(J;X)$ is uniformly integrable by (3). Moreover, the w_n satisfy $w_n(t) \in C(t) := F(t, \overline{\{u_k(t) : k \geq 1\}})$ and the sets $C(t)$ are weakly compact by Proposition 2.3. Therefore we may assume $w_n \rightharpoonup w$ in $L^1(J;X)$ due to Lemma 3.2. By Mazur's theorem there are $\overline{w}_n \in \operatorname{conv}\{w_k : k \geq n\}$ such that $\overline{w}_n \to w$ in $L^1(J;X)$, hence $\overline{w}_{n_k}(t) \to w(t)$ a.e. on J for some subsequence $(\overline{w}_{n_k})$. To conclude $w(t) \in F(t, u(t))$ a.e. on J we argue as follows. Let $t \in J$ be such that $w_n(t) \in F(t, u_n(t))$ for all $n \geq 1$ and $\overline{w}_{n_k}(t) \to w(t)$. Given $x^* \in X^*$ and $\epsilon > 0$, it follows that $x^*(w_n(t)) \in x^*(F(t, u(t))) + (-\epsilon, \epsilon)$ for all large n, hence the same inclusion holds for $x^*(\overline{w}_{n_k}(t))$ for all large k; notice that $x^* \circ F(t, \cdot)$ is usc with compact convex values. Hence $x^*(w(t)) \in x^*(F(t, u(t))$ for all $x^* \in X^*$ which implies $w(t) \in F(t, u(t))$, since F has closed convex values. Consequently $\mathrm{Sel}\,(u) \neq \emptyset$. In fact the same argument (with $u_n \in C(J;X)$ instead of step-functions) together with Proposition 2.3 also shows that $\mathrm{Sel} : C(J;X) \to 2^{L^1(J;X)} \setminus \emptyset$ is weakly usc with weakly compact values.

3. Let $S(t)$ denote the semigroup generated by $-A$ and let ψ be the solution of

$$\psi' = c(t)(1 + \psi) \text{ a.e. on } J, \quad \psi(0) = \max\{|S(t)u_0| : t \in J\}.$$

Then $K_0 = \{u \in C(J;X) : |u(t)| \leq \psi(t) \text{ on } J\}$ is closed bounded convex such that $G(K_0) \subset K_0$; notice that $u \in K_0$ and $v = \mathcal{S}w$ for $w \in \mathrm{Sel}\,(u)$ implies

$$|v(t)| \leq |S(t)u_0| + \int_0^t |w(s)| ds \leq \psi(0) + \int_0^t c(s)(1 + \psi(s)) ds = \psi(t) \text{ on } J.$$

Evidently $G(K) \subset K$ for $K := \overline{\operatorname{conv}}\, G(K_0)$, the latter set is compact convex in $C(J;X)$ by Lemma 3.1, and $G : K \to 2^K \setminus \emptyset$ by the previous step. We claim that the values of G are contractible. To see this, let $C = G(u)$ for some $u \in K$, fix $\bar{w} \in \mathrm{Sel}\,(u)$ and define $h : [0,1] \times C \to C$ by

$$h(s,v)(t) = \begin{cases} v(t) & \text{if } t \in [0, sa] \\ \bar{u}(t; sa, v(sa)) & \text{if } t \in (sa, a] \end{cases},$$

where $\bar{u}(\cdot\,; t_0, u_0)$ is the solution of $u' + Au \ni \bar{w}(t)$ on $[t_0, a]$, $u(t_0) = u_0$. Notice that h maps into C, since $v = \mathcal{S}w$ for some $w \in \mathrm{Sel}\,(u)$, hence $h(s,v) = \mathcal{S}\tilde{w}$ with $\tilde{w} := w\chi_{[0,sa]} + \bar{w}\chi_{(sa,a]} \in \mathrm{Sel}\,(u)$. Given $0 \leq s < \hat{s} \leq 1$ and $v, \hat{v} \in C$ let $\varphi(t) = |h(s,v)(t) - h(\hat{s}, \hat{v})(t)|$ on J. Then $\varphi(t) = |v(t) - \hat{v}(t)|$ on $[0, sa]$, $\varphi(t) \leq \varphi(s) + 2\int_s^t c(\tau)(1 + |u|_0) d\tau$ on $[sa, \hat{s}a]$ and $\varphi(t) \leq \varphi(\hat{s})$ on $[\hat{s}a, a]$ by the inequality for integral solutions. Therefore

$$|h(s,v) - h(\hat{s}, \hat{v})|_0 \leq |v - \hat{v}|_0 + 2(1 + |u|_0) \int_{sa}^{\hat{s}a} c(t) dt,$$

which yields continuity of h. Evidently $h(0,v) = \mathcal{S}\bar{w}$ and $h(1,v) = v$ on C, hence C is contractible.

Finally, $\mathrm{gr}\,(G)$ is a closed subset of $K \times K$, hence compact. Therefore G is usc by Proposition 2.1, and application of Lemma 2.1 yields a fixed point of G. $\square$

By means of Lemma 3.3 we are now able to obtain the first main result.

Theorem 3.1 *Let A be m-accretive in a real Banach space X such that $-A$ generates a compact semigroup. Let $J = [0,a] \subset \mathbb{R}$ and $F : J \times \overline{D(A)} \to 2^X \setminus \emptyset$ with closed convex values be such that $F(\cdot, x)$ has a strongly measurable selection for every $x \in \overline{D(A)}$, $F(t, \cdot)$ is weakly usc for almost all $t \in J$ and (3) is satisfied. Then (1) has a mild solution for every $u_0 \in \overline{D(A)}$, if also one of the following conditions is valid.*
(a) $F = \{f\}$ with $f : J \times \overline{D(A)} \to X$ continuous in x.
(b) X^* is uniformly convex.
(c) X^* is strictly convex, $F(t, \cdot)$ is usc with compact values for almost all $t \in J$.
(d) A is linear and densely defined, F has weakly compact values.

Proof. We only have to show that $\mathrm{gr}\,(G)$ is closed, since then Lemma 3.3 applies; notice that the values of F are weakly compact in each case. For this purpose, let $v_n \in G(u_n)$ with $u_n \to u$ and $v_n \to v$ in $C(J;X)$. Then $v_n = \mathcal{S}w_n$ with $w_n \in \mathrm{Sel}\,(u_n)$, and $|w_n(t)| \le \varphi(t)$ a.e. on J with $\varphi \in L^1(J)$ for all $n \ge 1$.

Now the single-valued case (a) is particularly simple, since $u_n \to u$ in $C(J;X)$ and $w_n \in \mathrm{Sel}\,(u_n)$ imply $w_n = f(\cdot, u_n(\cdot)) \to w = f(\cdot, u(\cdot))$ a.e. on J. Hence $w_n \to w$ in $L^1(J;X)$ by the dominated convergence theorem and therefore $v_n = \mathcal{S}w_n \to \mathcal{S}w$, i.e. $v = \mathcal{S}w \in G(u)$.
In all remaining cases we may assume $w_n \rightharpoonup w \in \mathrm{Sel}\,(u)$ in $L^1(J;X)$ by step 2 of the proof of Lemma 3.3. Let $\overline{v} = \mathcal{S}w$ and $z_n = w - w_n$.

If (b) holds, the inequality for integral solutions (see (3) in §1.4) implies

$$\frac{1}{2}|\overline{v}(t) - v_n(t)|^2 \le \int_0^t (z_n(s), \overline{v}(s) - v_n(s))_+ ds \quad \text{on } J \text{ for all } n \ge 1.$$

Now recall that the duality map $\mathcal{F} : X \to X^*$ is single-valued and uniformly continuous on bounded sets by Proposition 1.3 since X^* is uniformly convex. Hence

$$\frac{1}{2}|\overline{v}(t)-v_n(t)|^2 \le \int_0^t \mathcal{F}(\overline{v}(s)-v(s))(z_n(s))ds + |z_n|_1 \sup_J ||\mathcal{F}(\overline{v}(s)-v_n(s)) - \mathcal{F}(\overline{v}(s)-v(s))|| \to 0;$$

notice that (z_n) is bounded in $L^1(J;X)$ and that $z \to \int_0^t \mathcal{F}(\overline{v}(s) - v(s))(z(s))ds$ defines a continuous linear functional on $L^1(J;X)$. Consequently $v = \overline{v}$, hence $v = \mathcal{S}w \in G(u)$.

If (c) is satisfied, then $z_n \rightharpoonup 0$ in $L^1(J;X)$ implies

$$\overline{\lim_{n\to\infty}} \frac{1}{2}|\overline{v}(t) - v_n(t)|^2 \le \overline{\lim_{n\to\infty}} \int_0^t [\mathcal{F}(\overline{v}(s) - v_n(s)) - \mathcal{F}(\overline{v}(s) - v(s))](z_n(s))ds \quad \text{on } J.$$

For $s \in J$ let $x_n^*(s) = \mathcal{F}(\overline{v}(s) - v_n(s)) - \mathcal{F}(\overline{v}(s) - v(s))$. Since $\mathcal{F}$ is continuous from X with the norm-topology to X^* with the weak*-topology due to strict convexity of X^*, it follows that $x_n^*(s)(x) \to 0$ for all $x \in X$, uniformly on compact sets. Now recall that $w_n(t) \in C(t) := F(t, \overline{\{u_k(t) : k \ge 1\}})$ a.e. on J, and the $C(t) \subset X$ are compact by Proposition 2.1 due to the assumptions on $F(t,\cdot)$. Therefore $x_n^*(s)(z_n(s)) \to 0$ a.e. on J. Moreover, $|x_n^*(s)(z_n(s))| \leq M\varphi(s)$ a.e. on J with some $M > 0$. Hence $v = \overline{v} \in G(u)$ by the dominated convergence theorem.

If condition (d) holds then all v_n satisfy the variation of constants formula, i.e.

$$v_n(t) = S(t)u_0 + \int_0^t S(t-s)w_n(s)ds \quad \text{on } J,$$

where $S(t)$ denotes the semigroup generated by $-A$; see Proposition 1.5. Given $x^* \in X^*$, it follows that

$$x^*(\overline{v}(t) - v_n(t)) = \int_0^t x^*(S(t-s)(w_n(s) - w(s))ds \to 0 \quad \text{as } n \to \infty,$$

hence $\overline{v}(t) = v(t)$ on J and therefore $v = \overline{v} \in G(u)$. □

3.3 Perturbations of dissipative type.

In the present section we first consider single-valued perturbations, i.e.

$$u' + Au \ni f(t,u) \quad \text{on } J, \quad u(0) = u_0 \tag{4}$$

with $J = [0,a]$, where we assume that $f : J \times X \to X$ is *Carathéodory*, i.e. f is strongly measurable in t and continuous in x, and satisfies the dissipativity condition

$$(f(t,x) - f(t,\overline{x}), x - \overline{x})_- \leq k(t)|x - \overline{x}|^2 \quad \text{for a.a. } t \in J, \text{ all } x, \overline{x} \in X \text{ with } k \in L^1(J). \tag{5}$$

Existence of a mild solution will be obtained by reduction to the case of jointly continuous f, which is based on

Lemma 3.4 *Let X be a separable Banach space, $K \subset X$ closed and $f : J \times K \to X$ with $J = [0,a] \subset \mathbb{R}$ be Carathéodory. Then f is almost continuous, i.e. for every $\epsilon > 0$ there exists a closed $J_\epsilon \subset J$ with $\lambda_1(J \setminus J_\epsilon) \leq \epsilon$ such that $f_{|J_\epsilon \times K}$ is continuous.*

Lemma 3.4 is a special case of Theorem 2 in Kucia [74].
After reduction to continuous f the following basic existence result applies.

Lemma 3.5 *Let A be m-accretive in a real Banach space X, $f : J \times X \to X$ with $J = [0,a] \subset \mathbb{R}$ be continuous and such that (5) holds with $k(t) \equiv k$. Then (4) has a unique mild solution for every $u_0 \in \overline{D(A)}$.*

For $k = 0$ this is Theorem III.3.1 in Barbu [14]. For general k see Theorem III in Pierre [94], where it is actually shown that Lemma 3.5 remains true for continuous $f : J \times \overline{D(A)} \to X$.

The next result provides an extension to the Carathéodory case under the additional growth condition

$$|f(t,x)| \le c(t)(1+|x|) \quad \text{on } J \times X \text{ with some } c \in L^1(J); \tag{6}$$

notice that some assumption concerning the growth of f is needed, since $f(t,x) = g(t)$ is Carathéodory and satisfies (5) with $k=0$ whenever $g : J \to X$ is strongly measurable.

Theorem 3.2 *Let A be m-accretive in a real Banach space X, $f : J \times X \to X$ with $J = [0,a] \subset \mathbb{R}$ be Carathéodory satisfying (5) and (6). Then (4) has a unique mild solution for every $u_0 \in \overline{D(A)}$.*

Proof. Since almost all $f(t,\cdot)$ are continuous and everywhere defined, condition (5) implies the same inequality with $(\cdot,\cdot)_-$ replaced by $(\cdot,\cdot)_+$; see Proposition 1.2. Hence uniqueness of mild solutions is a direct consequence of the inequality (3) in §1.4 for integral solutions.

1. We first reduce to the case of separable X as follows. We claim that there is a closed separable subspace $\hat{X} \subset X$ with $u_0 \in \hat{X}$ and a measurable $\hat{J} \subset J$ with $\lambda_1(J \setminus \hat{J}) = 0$ such that $f(\hat{J} \times \hat{X}) \subset \hat{X}$ and $(I+\lambda A)^{-1}\hat{X} \subset \hat{X}$ for all $\lambda > 0$.

If this holds, the restriction $\hat{A}$ of A to $\hat{X}$ (defined by means of $\mathrm{gr}\,(\hat{A}) = \mathrm{gr}\,(A) \cap [\hat{X} \times \hat{X}]$) is m-accretive in $\hat{X}$. Since the solution set of (4) is not changed if we redefine f to be zero for all $t \in J \setminus \hat{J}$, it suffices to consider (4) in the separable Banach space $\hat{X}$ then.

To prove the claim, let $X_0 = \mathrm{span}\{u_0\}$ and $M_0 \subset X_0$ a countable dense subset of X_0. Since $f(\cdot,x)$ is strongly measurable for all $x \in M_0$, there is a measurable subset J_0 of J with $\lambda_1(J \setminus J_0) = 0$ such that $f(J_0 \times M_0)$ is contained in a separable subspace. In addition, J_0 can be chosen such that $f(t,\cdot)$ is continuous for all $t \in J_0$. Let

$$X_1 = \overline{\mathrm{span}}\Big(X_0 \cup f(J_0 \times X_0) \cup \bigcup_{\lambda>0} (I+\lambda A)^{-1}X_0\Big).$$

To see that X_1 is separable, recall that $\overline{\mathrm{span}}\,\overline{M} = \overline{\mathrm{span}}\,M$ for any $M \subset X$ and notice that $f(J_0 \times X_0) \subset \overline{f(J_0 \times M_0)}$ by continuity of the $f(t,\cdot)$, and

$$\bigcup_{\lambda>0} (I+\lambda A)^{-1}X_0 \subset \overline{\{(I+\lambda A)^{-1}x : x \in M_0, 0 < \lambda \in \mathbb{Q}\}}.$$

By induction, we get a decreasing sequence of measurable $J_n \subset J$ with $\lambda_1(J \setminus J_n) = 0$ and an increasing sequence of separable subspaces X_n such that

$$X_{n+1} = \overline{\mathrm{span}}\Big(X_n \cup f(J_n \times X_n) \cup \bigcup_{\lambda>0} (I+\lambda A)^{-1}X_n\Big).$$

Let $\hat{J} = \bigcap_{n\ge0} J_n$ and $\hat{X} = \overline{\bigcup_{n\ge0} X_n}$. Evidently $\hat{J} \subset J$ is measurable with $\lambda_1(J \setminus \hat{J}) = 0$ and $\hat{X}$ is a closed separable subspace. Given $(t,x) \in \hat{J} \times \hat{X}$, there is a sequence (x_k) with $x_k \to x$

and $x_k \in X_{n_k}$ for some $n_k \geq 0$. Hence $y_k := f(t, x_k) \to f(t, x)$ and $y_k \in X_{n_k+1} \subset \bigcup_{n \geq 0} X_n$ implies $f(t, x) \in \hat{X}$. The same argument yields $(I + \lambda A)^{-1} x \in \hat{X}$ for any $\lambda > 0$, hence the claim holds.

2. By the previous step we may assume that X is separable, hence f is almost continuous by Lemma 3.4. For fixed $\epsilon > 0$ this yields approximate solutions by reduction to the continuous case as follows. Let $J_\epsilon \subset J$ be closed with $\lambda_1(J \setminus J_\epsilon) \leq \epsilon$ such that $f_{|J_\epsilon \times X}$ is continuous, where we may also assume that $c_{|J_\epsilon}$, $k_{|J_\epsilon}$ are continuous and $\{0, a\} \subset J_\epsilon$. Since $J \setminus J_\epsilon \subset \mathbb{R}$ is an open set, there are disjoint $(a_n, b_n) \subset J$ such that $J \setminus J_\epsilon = \bigcup_{n \geq 1} (a_n, b_n)$. Define $f_\epsilon : J \times X \to X$ by

$$f_\epsilon(t, x) = \begin{cases} f(t, x) & \text{if } t \in J_\epsilon \\ \dfrac{b_n - t}{b_n - a_n} f(a_n, x) + \dfrac{t - a_n}{b_n - a_n} f(b_n, x) & \text{if } t \in (a_n, b_n). \end{cases} \tag{7}$$

Evidently f_ϵ is continuous and satisfies (5) with $k_\epsilon = \max_{J_\epsilon} k(t)$ in place of $k(t)$; recall that the stronger version of (5) with $(\cdot, \cdot)_+$ holds for f. By means of Lemma 3.5 there is a mild solution of (4) with f replaced by f_ϵ.

Consider $\epsilon_n \searrow 0$ with $\sum_{n \geq 1} \epsilon_n < \infty$. For $n \geq 1$ let f_n be given by (7) for $\epsilon = \epsilon_n$, let J_n denote the corresponding set J_{ϵ_n} and u_n be a mild solution of (4) with f_n instead of f. Suppose that the J_n can be chosen in such a way that, in addition, $J_n \subset J_{n+1}$ and

$$|f_n(t, x)| \leq \hat{c}(t)(1 + |x|) \text{ on } J \times X \text{ for all } n \geq 1 \text{ with } \hat{c} \in L^1(J). \tag{8}$$

If this holds then

$$|u_n(t)| \leq |S(t) u_0| + \int_0^t \hat{c}(s)(1 + |u_n(s)|) ds \text{ on } J,$$

hence there is $R > 0$ such that $|u_n|_0 \leq R$ for all $n \geq 1$. Now consider $m, n \geq p$ for fixed $p \geq 1$. By definition of f_n we have $f_n(t, x) = f_m(t, x) = f(t, x)$ on $J_p \times X$, hence

$$|u_n(t) - u_m(t)|^2 \leq 2 \int_0^t (f_n(s, u_n(s)) - f_m(s, u_m(s)), u_n(s) - u_m(s))_+ ds$$

$$\leq 2 \int_{[0,t] \cap J_p} k(s) |u_n(s) - u_m(s)|^2 ds + 4R \int_{[0,t] \setminus J_p} |f_n(s, u_n(s)) - f_m(s, u_m(s))| ds \text{ on } J.$$

This implies

$$|u_n(t) - u_m(t)|^2 \leq 2 \int_0^t k(s) |u_n(s) - u_m(s)|^2 ds + \delta_p \text{ on } J$$

with

$$\delta_p \leq 8R(1 + R) \int_{J \setminus J_p} \hat{c}(s) ds \to 0 \text{ as } p \to \infty.$$

Application of Gronwall's lemma shows that $|u_n - u_m|_0 \leq \sqrt{\delta_p} e^{|k|_1}$ for $m, n \geq p$, hence (u_n) is Cauchy in $C(J; X)$, i.e. $|u_n - u|_0 \to 0$ for some $u \in C(J; X)$. Since $f_n(t, u_n(t)) \to f(t, u(t))$

a.e. on every J_p, hence a.e. on J, and $|f_n(t,u_n(t))| \le \hat{c}(t)(1+R)$ a.e. on J, the dominated convergence theorem yields $w_n := f_n(\cdot,u_n(\cdot)) \to w := f(\cdot,u(\cdot))$ in $L^1(J;X)$. Therefore $u_n = \mathcal{S}w_n \to \mathcal{S}w$, i.e. $u = \mathcal{S}w$ is a mild solution of (4).

To show that the J_n can be chosen in the way described above, let

$$M_l = \{t \in J : l-1 \le c(t) < l\} \quad \text{for } l \ge 1.$$

For every $n \ge 1$ we then find $m_n \ge 2$ such that $\tilde{J}_n := \bigcup_{l=1}^{m_n} M_l$ satisfies $\lambda_1(J \setminus \tilde{J}_n) \le \epsilon_n/2$, and to every $l = 1,\dots,m_n$ there is a closed $B_l (= B_{l,n}) \subset M_l$ with $\lambda_1(M_l \setminus B_l) \le \dfrac{\epsilon_n}{m_n^2}$ such that $f_{|B_l \times X}$, $c_{|B_l}$ and $k_{|B_l}$ are continuous. Let $J_n = \bigcup_{l=1}^{m_n} B_l$ and $N_n = \tilde{J}_n \setminus J_n$. Then $\lambda_1(N_n) \le \epsilon_n/m_n$, hence $\lambda_1(J \setminus J_n) \le \epsilon_n$, and J_n has the properties required above. Given J_n, we choose $m_{n+1} \ge m_n$ and $B_{l,n+1} \supset B_{l,n}$ for $l = 1,\dots,m_n$ to get $J_n \subset J_{n+1}$ for all $n \ge 1$.
Due to $c(t) < m_n$ on $\tilde{J}_n \supset J_n$ and $c(t) \ge m_n$ on $J \setminus \tilde{J}_n$, the definition of f_n shows that $|f_n(t,x)| \le c(t)(1+|x|)$ for $t \in J_n \cup (J \setminus \tilde{J}_n)$ and $|f_n(t,x)| \le m_n(1+|x|)$ for $t \in N_n$. Hence all f_n satisfy (8) with $\hat{c}(t) := c(t) + \sum_{n \ge 1} m_n \chi_{N_n}(t)$ on J, and $|\hat{c}|_1 \le |c|_1 + \sum_{n\ge 1} \epsilon_n < \infty$. □

Let us now consider multivalued perturbations of usc type under the condition

$$(y-\overline{y}, x-\overline{x})_+ \le k(t)|x-\overline{x}|^2 \quad \text{for a.a. } t \in J, \text{ all } x,\overline{x} \in X, y \in F(t,x), \overline{y} \in F(t,\overline{x}) \tag{9}$$

with some $k \in L^1(J)$.

Theorem 3.3 *Let X be a real Banach space with uniformly convex dual and A be m-accretive in X. Let $J = [0,a] \subset \mathbb{R}$ and $F : J \times X \to 2^X \setminus \emptyset$ with closed convex values satisfying (3) and (9) be such that all $F(\cdot,x)$ have a strongly measurable selection and $F(t,\cdot)$ is weakly usc for almost all $t \in J$. Then (1) has a unique mild solution for every $u_0 \in \overline{D(A)}$.*

Proof. 1. Given $(r_n) \subset (0,1]$ with $r_n \searrow 0$, we approximate F by single-valued f_n as follows. Fix $n \ge 1$, let $(U_\lambda)_{\lambda\in\Lambda}$ be a locally finite refinement of the open covering $X = \bigcup_{x \in X} B_{r_n}(x)$ and $(\varphi_\lambda)_{\lambda\in\Lambda}$ be a locally Lipschitz partition of unity subordinate to $(U_\lambda)_{\lambda\in\Lambda}$. For every $\lambda \in \Lambda$ choose $x_\lambda \in X$ such that $U_\lambda \subset B_{r_n}(x_\lambda)$, let $g_\lambda : J \to X$ be a strongly measurable selection of $F(\cdot,x_\lambda)$ and define f_n by

$$f_n(t,x) = \sum_{\lambda \in \Lambda} \varphi_\lambda(x) g_\lambda(t) \quad \text{on } J \times X.$$

All f_n are strongly measurable in t, hence also Carathéodory, satisfying

$$f_n(t,x) \in \overline{\text{conv}}\, F(t, B_{r_n}(x)) \quad \text{and} \quad |f_n(t,x)| \le c(t)(2+|x|) \quad \text{on } J \times X. \tag{10}$$

Indeed, $\varphi_\lambda(x) > 0$ for some $\lambda \in \Lambda$ implies $x \in U_\lambda \subset B_{r_n}(x_\lambda)$ and therefore $g_\lambda(t) \in F(t, B_{r_n}(x))$ as well as $|g_\lambda(t)| \le c(t)(1 + r_n + |x|)$.
Moreover, for every $x \in X$ there is $\delta = \delta_n(x) > 0$ and $L = L_n(x) > 0$ such that

$$|f_n(t,y) - f_n(t,\overline{y})| \le c(t)L|y - \overline{y}| \quad \text{for all } t \in J \text{ and } y, \overline{y} \in \overline{B}_\delta(x). \tag{11}$$

To obtain (11), let $\eta > 0$ be such that $B_\eta(x)$ intersects only finitely many U_λ, and $\delta \in (0, \eta)$ be such that the corresponding φ_λ are Lipschitz continuous on $\overline{B}_\delta(x)$.

2. Fix $n \ge 1$ and consider the initial value problem

$$u' + Au \ni f_n(t,u) \ \text{ on } [t_0, a], \quad u(t_0) = u_0 \tag{12}$$

for $t_0 \in [0, a)$ and $u_0 \in \overline{D(A)}$. To get a local mild solution of (12), let $\delta, L > 0$ be such that (11) holds on $\overline{B}_\delta(u_0)$, and define f by $f(t,x) = f_n(t, Rx)$ on $J \times X$, where R is the usual retraction onto $\overline{B}_\delta(u_0)$, i.e.

$$Rx = \begin{cases} x & \text{if } |x - u_0| \le \delta \\ u_0 + \delta \dfrac{x - u_0}{|x - u_0|} & \text{if } |x - u_0| > \delta. \end{cases}$$

Then f is Carathéodory and satisfies (5) with $k(t) = 2Lc(t)$ since R is Lipschitz continuous of constant 2. Hence Theorem 3.2 yields a solution u of (12) with f instead of f_n, and u is obviously a solution of (12) on $[t_0, t_0 + h]$ for some $h > 0$.
Consequently, having (unique) local solvability of (12), Zorn's lemma yields a noncontinuable mild solution u of (12) with $t_0 = 0$, which is defined on all of J or on $[0, \tau)$ with some $\tau \le a$. In the latter case we get a contradiction since then $\lim_{t \to \tau-} u(t)$ exists. Indeed,

$$|u(t)| \le |S(t)u_0| + \int_0^t c(s)(2 + |u(s)|)ds \ \text{ on } [0, \tau)$$

by (10), hence $|u(t)| \le R$ and $|f_n(t, u(t))| \le c(t)(2 + R)$ on $[0, \tau)$ with some $R > 0$. This shows that $w \in L^1(J; X)$, if we let $w(t) = f_n(t, u(t))$ on $[0, \tau)$ and $w(t) = 0$ on $[\tau, a]$. Hence (2) has mild solution $\mathcal{S}w$ on J and $(\mathcal{S}w)(t) = u(t)$ on $[0, \tau)$, i.e. $\lim_{t \to \tau-} u(t) = (\mathcal{S}w)(\tau)$.

3. By the previous steps, initial value problem (12) with $t_0 = 0$ has a mild solution u_n and $|u_n|_0 \le R$ for all $n \ge 1$. We claim that (u_n) is Cauchy in $C(J; X)$.
Let $\epsilon > 0$. Since the duality map $\mathcal{F} : X \to X^*$ is uniformly continuous on bounded sets, there is $\delta > 0$ such that $||\mathcal{F}(x) - \mathcal{F}(\overline{x})|| \le \epsilon$ for all $x, \overline{x}$ with $|x|, |\overline{x}| \le R + 1$ and $|x - \overline{x}| \le \delta$. To obtain an estimate for $|u_n(t) - u_m(t)|$, notice first that $z \in \overline{\text{conv}} F(t, B_\rho(x))$ and $\overline{z} \in \overline{\text{conv}} F(t, B_{\overline{\rho}}(\overline{x}))$ for $|x|, |\overline{x}| \le R$ and $\rho, \overline{\rho} \le 1$ with $\rho + \overline{\rho} \le \delta$ imply

$$(z - \overline{z}, x - \overline{x}) \le 2k(t)((\rho + \overline{\rho})^2 + |x - \overline{x}|^2) + 2(2 + R)c(t)\epsilon;$$

here and in the sequel we write $(\cdot,\cdot)$ instead of $(\cdot,\cdot)_+$ since the semi-inner products coincide. It suffices to consider $z \in \mathrm{conv}F(t, B_\rho(x))$ and $\overline{z} \in \mathrm{conv}F(t, B_{\overline{\rho}}(\overline{x}))$, hence

$$z = \sum_{i=1}^{p} \lambda_i y_i \quad \text{with } \lambda_i > 0 \text{ such that } \sum_{i=1}^{p} \lambda_i = 1, \quad y_i \in F(t, x_i) \text{ with } x_i \in B_\rho(x),$$
$$\overline{z} = \sum_{j=1}^{q} \mu_j \overline{y}_j \quad \text{with } \mu_j > 0 \text{ such that } \sum_{j=1}^{q} \mu_j = 1, \quad \overline{y}_j \in F(t, \overline{x}_j) \text{ with } \overline{x}_j \in B_{\overline{\rho}}(\overline{x}).$$

Then

$$(z - \overline{z}, x - \overline{x}) = \sum_{i=1}^{p}\sum_{j=1}^{q} \lambda_i \mu_j (y_i - \overline{y}_j, x - \overline{x})$$
$$\leq \sum_{i=1}^{p}\sum_{j=1}^{q} \lambda_i \mu_j (y_i - \overline{y}_j, x_i - \overline{x}_j) + \sum_{i=1}^{p}\sum_{j=1}^{q} \lambda_i \mu_j |y_i - \overline{y}_j| \epsilon$$
$$\leq \sum_{i=1}^{p}\sum_{j=1}^{q} \lambda_i \mu_j k(t) |x_i - \overline{x}_j|^2 + c(t)(2 + |x| + |\overline{x}| + \rho + \overline{\rho})\epsilon$$
$$\leq k(t)(|x - \overline{x}| + \rho + \overline{\rho})^2 + 2c(t)(2 + R)\epsilon,$$

hence the inequality above holds. By means of this estimate we obtain

$$|u_n(t) - u_m(t)|^2 \leq 2 \int_0^t (f_n(s, u_n(s)) - f_m(s, u_m(s)), u_n(s) - u_m(s)) ds$$
$$\leq 4 \int_0^t k(s) |u_n(s) - u_m(s)|^2 ds + 4(r_n + r_m)^2 |k|_1 + 2(2 + R)|c|_1 \epsilon$$

for all large m, n. Application of Gronwall's lemma shows that (u_n) is Cauchy in $C(J; X)$, hence $u_n \to u$ for some $u \in C(J; X)$.

To finish the proof let $w_n = f_n(\cdot, u_n(\cdot))$, hence $u_n = \mathcal{S}w_n \to u$. Since $|w_n(t)| \leq c(t)(2+R)$ a.e. on J for all $n \geq 1$ and X is reflexive, we may assume $w_n \rightharpoonup w$ in $L^1(J; X)$ due to Lemma 3.2, and then $\mathcal{S}w = u$ follows as in the proof to Theorem 3.1(b). Therefore, it remains to show $w \in \mathrm{Sel}\,(u)$. Given $\eta > 0$, we have $w_n(t) \in \overline{\mathrm{conv}}F(t, B_\eta(u(t)))$ a.e. on J for all large n by (10), hence $w(t) \in \overline{\mathrm{conv}}F(t, B_\eta(u(t)))$ a.e. on J for every $\eta > 0$. Fix $t \in J$ such that $F(t, \cdot)$ is weakly usc and the last inclusion holds, let $x^* \in X^*$ and notice that $x^*(\overline{\mathrm{conv}}K) \subset \overline{\mathrm{conv}}\, x^*(K)$ for every $K \subset X$. Given $\epsilon > 0$, it follows that

$$x^*(w(t)) \in x^*[\overline{\mathrm{conv}}F(t, B_\eta(u(t)))] \subset \overline{\mathrm{conv}}[x^*(F(t, u(t))) + (-\epsilon, \epsilon)]$$

for sufficiently small $\eta > 0$, since $x^* \circ F(t, \cdot)$ is usc with compact convex values. This implies $x^*(w(t)) \in x^*(F(t, u(t)))$ for every $x^* \in X^*$, hence $w(t) \in F(t, u(t))$. Consequently $w \in \mathrm{Sel}\,(u)$. $\square$

3.4 Perturbations of compact type.

In the important special case when X is a Hilbert space and $A = \partial\varphi$ is the subdifferential of a proper convex lsc function $\varphi : D_\varphi \subset X \to \mathbb{R}$, the semigroup generated by $-A$ is always equicontinuous and, in this situation, the semigroup is compact iff φ has compact sublevel sets, i.e. $\{x \in X : |x|^2 + \varphi(x) \le r\}$ is compact for all $r > 0$; see e.g. p.42f in Vrabie [112]. This is one motivation to consider initial value problem (1) in case the semigroup generated by $-A$ is only equicontinuous. Let us also mention that $-A$ generates an equicontinuous semigroup if A is m-accretive and homogeneous of degree $\alpha > 0$, $\alpha \ne 1$; this is a direct consequence of Theorem 1 in Benilan/Crandall [16].

Instead of compactness of the resolvents of A we then impose a compactness condition on F. More precisely, we assume that $F : J \times D \to 2^X \setminus \emptyset$ satisfies

$$\beta(F(t,B)) \le k(t)\beta(B) \ \text{ a.e. on } J \text{ for all bounded } B \subset D \text{ with } k \in L^1(J), \tag{13}$$

where $\beta(\cdot)$ denotes the Hausdorff-measure of noncompactness introduced in §2.3. To obtain existence of a mild solution within this situation, the fixed point approach is useful again but this time it is harder to find a compact (convex) $K \subset C(J;X)$ such that $G(K) \subset K$ for $G = \mathcal{S} \circ \mathrm{Sel}$. If $W \subset L^1(J;X)$ is uniformly integrable then, directly by means of equicontinuity of the semigroup, only equicontinuity of $\mathcal{S}(W)$ follows. This is the first part in

Lemma 3.6 *Let A be m-accretive in a real Banach space X such that $-A$ generates an equicontinuous semigroup.*
(a) Let $W \subset L^1(J;X)$ be uniformly integrable. Then $\mathcal{S}(W) \subset C(J;X)$ is equicontinuous.
(b) Let X^* be uniformly convex, $C \subset X$ compact and let $W = \{w \in L^1(J;X) : w(t) \in C$ a.e. on $J\}$. Then $\mathcal{S}(W)$ is relatively compact in $C(J;X)$.

This is essentially Theorem 2.3, respectively Theorem 3.1(ii) in Gutman [64]. Now the point is whether (13) implies relative compactness of the sections $\{(\mathcal{S}w)(t) : w \in W\}$. In case $A = 0$ this can be achieved by means of the following estimate; see Proposition 9.3 in Deimling [42].

Proposition 3.1 *Let X be a separable Banach space, $J = [0,a] \subset \mathbb{R}$ and $(w_k) \subset L^1(J;X)$ such that $|w_k(t)| \le \varphi(t)$ a.e. on J for all $k \ge 1$ with some $\varphi \in L^1(J)$. Then*

$$\beta(\{\int_0^t w_k(s)\,ds : k \ge 1\}) \le \int_0^t \beta(\{w_k(s) : k \ge 1\})\,ds \quad \textit{on } J. \tag{14}$$

Now the idea is to extend this estimate to the case $A \ne 0$. More precisely, we will show that if X^* is uniformly convex then

$$\beta(\{(\mathcal{S}w_k)(t) : k \ge 1\}) \le \int_0^t \beta(\{w_k(s) : k \ge 1\})\,ds \quad \text{ on } J. \tag{15}$$

This needs some preparation. Given $\emptyset \neq \Omega \subset X$, recall that $\beta_\Omega(B)$ is defined by

$$\beta_\Omega(B) = \inf\{r > 0 : B \subset \bigcup_{i=1}^m B_r(x_i) \text{ for some } m \geq 1 \text{ and } x_1, \ldots, x_m \in \Omega\}$$

for bounded $B \subset \Omega$, i.e. the centers of the covering balls are chosen from Ω instead of X. Then $\beta(B) \leq \beta_\Omega(B) \leq 2\beta(B)$ for all bounded $B \subset \Omega$. Moreover, β_Ω has the following representation.

Proposition 3.2 *Let X be a Banach space and $\emptyset \neq \Omega_n \subset X$ with $\Omega_n \subset \Omega_{n+1}$ for $n \geq 1$ be such that $\beta(\Omega_n \cap A) = 0$ for bounded $A \subset X$ and all $n \geq 1$. Let $\Omega = \overline{\bigcup_{n\geq 1} \Omega_n}$ and $B = \{x_k : k \geq 1\} \subset \Omega$ be bounded. Then $\beta_\Omega(B) = \lim_{n\to\infty} \overline{\lim_{k\to\infty}}\, \rho(x_k, \Omega_n)$.*

This is an extension of Proposition 9.2 in Deimling [42], where $\Omega := X$ is assumed to be separable and the Ω_n are subspaces of finite dimension. Nevertheless, except for trivial modifications, the proof given there also works in the situation considered here. Now we have

Lemma 3.7 *Let X be a real Banach space with uniformly convex dual and A be m-accretive in X such that $-A$ generates an equicontinuous semigroup. Let $J = [0, a] \subset \mathbb{R}$ and $(w_k) \subset L^1(J; X)$ such that $|w_k(t)| \leq \varphi(t)$ a.e. on J for all $k \geq 1$ with some $\varphi \in L^1(J)$.*
Then (15) holds.

Proof. We may assume that $X_0 = \overline{\text{span}}\{w_k(t) : t \in J, k \geq 1\}$ is separable, since all w_k are strongly measurable. By Theorem V.2.3 in Diestel [47], which applies since X is in particular reflexive, there is a closed separable subspace Y of X, containing X_0, and a linear continuous projection P from X onto Y with $\|P\| = 1$. For bounded $B \subset Y$ we therefore have $\beta(B) = \beta_Y(B)$. Let $Y_n \subset Y$ be finite-dimensional subspaces such that $Y = \overline{\bigcup_{n\geq 1} Y_n}$,

$$W_n = \{w \in L^1(J; Y_n) : |w(s)| \leq 2\varphi(s) \;\text{ a.e. on } J\}$$

and

$$\Omega_n = \{(\mathcal{S}w)(t) : w \in W_n\} \quad \text{for fixed } t \in J.$$

We claim that $\beta(\Omega_n) = 0$ for all $n \geq 1$, where it suffices to consider Ω_n for $t > 0$. For every $\epsilon > 0$ there is a closed $J_\epsilon \subset J$ such that $\varphi_{|J_\epsilon}$ is continuous (hence also bounded) and $\int_{J\setminus J_\epsilon} \varphi(t)\, dt \leq \epsilon/2$. Then

$$\Omega_n^\epsilon := \{(\mathcal{S}v)(t) : v = w\chi_{J_\epsilon} \quad \text{with } w \in W_n\}$$

is relatively compact by Lemma 3.6. Since $x := (\mathcal{S}w)(t) \in \Omega_n$ implies $x^\epsilon := (\mathcal{S}(w\chi_{J_\epsilon}))(t) \in \Omega_n^\epsilon$ and $|x - x_\epsilon| \leq \int_{J\setminus J_\epsilon} |w(t)|\, dt \leq \epsilon$, we have $\Omega_n \subset \Omega_n^\epsilon + B_\epsilon(0)$. This yields $\beta(\Omega_n) \leq \beta(\Omega_n^\epsilon) + \epsilon = \epsilon$ for all $\epsilon > 0$, i.e. $\beta(\Omega_n) = 0$.

Let $\Omega = \overline{\bigcup_{n\geq 1} \Omega_n}$ and notice that $(\mathcal{S}w_k)(t) \in \Omega$ for all $k \geq 1$. To see this, fix $k \geq 1$ and define the multivalued map $H_n : J \to 2^{Y_n}$ by

$$H_n(s) := \{x \in Y_n : |w_k(s) - x| \leq \rho(w_k(s), Y_n)\} \quad \text{for } s \in J.$$

Evidently H_n has nonempty closed values, and H_n is measurable since w_k has this property. Therefore H_n admits a measurable selection v_n by Lemma 2.2, and $|v_n(s)| \leq 2\varphi(s)$ a.e. on J. Moreover, $v_n \to w_k$ in $L^1(J;X)$ as $n \to \infty$ since $w_k(s) \in Y$ a.e. on J. Hence $(\mathcal{S}v_n)(t) \in \Omega_n$ with $(\mathcal{S}v_n)(t) \to (\mathcal{S}w_k)(t)$ implies $(\mathcal{S}w_k)(t) \in \Omega$.
Consequently, Proposition 3.2 applies and yields

$$\beta(\{(\mathcal{S}w_k)(t) : k \geq 1\}) \leq \beta_\Omega(\{(\mathcal{S}w_k)(t) : k \geq 1\}) = \lim_{n\to\infty} \overline{\lim_{k\to\infty}} \rho((\mathcal{S}w_k)(t), \Omega_n).$$

Now

$$\begin{aligned} \rho((\mathcal{S}w_k)(t), \Omega_n) &= \inf\{|(\mathcal{S}w_k)(t) - (\mathcal{S}w)(t)| : w \in W_n\} \\ &\leq \inf\{\int_0^t |w_k(s) - w(s)|\, ds : w \in W_n\} = \int_0^t \rho(w_k(s), Y_n)\, ds, \end{aligned}$$

where the last equality follows by the special choice $w = v_n$ with the measurable selection v_n from above. Finally, application of Fatou's Lemma, the dominated convergence theorem and Proposition 3.2 (with Y instead of Ω) yields

$$\begin{aligned} \beta(\{(\mathcal{S}w_k)(t) : k \geq 1\}) &\leq \int_0^t \lim_{n\to\infty} \overline{\lim_{k\to\infty}} \rho(w_k(s), Y_n)\, ds \\ &= \int_0^t \beta_Y(\{w_k(s) : k \geq 1\})\, ds = \int_0^t \beta(\{w_k(s) : k \geq 1\})\, ds. \end{aligned}$$

$\square$

By means of Lemma 3.7 we are able to obtain the following existence result for perturbations of compact type.

Theorem 3.4 *Let X be a real Banach space with uniformly convex dual and A be m-accretive in X such that $-A$ generates an equicontinuous semigroup. Let $D = \overline{\text{conv}}\, D(A)$, $J = [0, a] \subset \mathbb{R}$ and $F : J \times D \to 2^X \setminus \emptyset$ with closed convex values satisfying (3) and (13) be such that $F(\cdot, x)$ has a strongly measurable selection for every $x \in D$ and $F(t, \cdot)$ is weakly usc for almost all $t \in J$. Then (1) has a mild solution for every $u_0 \in \overline{D(A)}$.*

Proof. We look for a fixed point of $G = \mathcal{S} \circ \text{Sel}$, and get a closed bounded convex set $K_0 \subset C(J;X)$ such that $G(K_0) \subset K_0$ by means of (3) as in the proof of Lemma 3.3. Let $K_{n+1} := \overline{\text{conv}}\, G(K_n)$ for $n \geq 0$ and $K := \bigcap_{n\geq 0} K_n$. We are done if K is relatively compact since then K is compact convex, and $G : K \to 2^K \setminus \emptyset$ is usc with closed contractible values which follows along the same line as in the proofs of Lemma 3.3 and Theorem 3.3(b). Consequently Lemma 2.1 then yields a fixed point of G, i.e. a mild solution of (1).

Application of Lemma 3.6 shows that K is an equicontinuous subset of $C(J;X)$, hence K is relatively compact if the sections $K(t) = \{u(t) : u \in K\}$ satisfy $\beta(K(t)) = 0$ on J. Let $\rho(t) = \beta(K(t))$ and $\rho_n(t) = \beta(K_n(t))$ for $n \geq 0$. Then $\rho_{n+1}(t) \leq \beta(\{(\mathcal{S}w)(t) : w \in \mathrm{Sel}\,(K_n)\})$. In order to apply Lemma 3.7 suppose, for the moment, that there is a sequence $(w_k) \subset \mathrm{Sel}\,(K_n)$ such that $\beta(\{(\mathcal{S}w)(t) : w \in \mathrm{Sel}\,(K_n)\}) \leq 3\beta(\{(\mathcal{S}w_k)(t) : k \geq 1\})$. Under this assumption it follows by (13) and (15) that

$$\rho_{n+1}(t) \leq 3\int_0^t \beta(\{w_k(s) : k \geq 1\})ds \leq 3\int_0^t k(s)\rho_n(s)ds \;\text{ on } J;$$

notice that the last integral makes sense since all ρ_n are in fact continuous due to equicontinuity of the K_n. Evidently $\rho_n(t) \searrow \rho_\infty(t)$ on J, hence

$$0 \leq \rho_\infty(t) \leq 2\int_0^t k(s)\rho_\infty(s)\,ds \;\text{ on } J, \quad \rho_\infty(0) = 0.$$

The same inequality holds with $r(t) = \displaystyle\int_0^t k(s)\rho_\infty(s)ds$ instead of $\rho_\infty(t)$, hence $r(t) = 0$ on J by Gronwall's lemma. This yields $\rho_\infty = 0$, hencen $\rho = 0$ follows from $0 \leq \rho(t) \leq \rho_\infty(t)$ on J.

To finish the proof it remains to show that for bounded $B \subset X$ there exists a sequence $(x_k) \subset B$ such that $\beta(B) \leq 3\beta(\{x_k : k \geq 1\})$, and it suffices to consider $B \subset X$ with $\beta(B) > 0$. Let $\rho = \beta(B)/3$, $r = \beta(B) - \rho$ and $x_1 \in B$. Then there is $x_2 \in B \setminus B_r(x_1)$ since otherwise $B \subset B_r(x_1)$ gives the contradiction $\beta(B) \leq r$. Given $x_1, \ldots, x_m \in B$ with $|x_j - x_k| \geq r$ for $j \neq k$, the same argument yields $x_{m+1} \in B$ such that $|x_j - x_{m+1}| \geq r$ for $j = 1, \ldots, m$. By induction we therefore get a sequence $(x_k) \subset B$ with $|x_j - x_k| \geq r$ for all $j \neq k$, which evidently implies $\beta(\{x_k : k \geq 1\}) \geq r/2$. Therefore $\beta(B) = r + \rho \leq 2\beta(\{x_k : k \geq 1\}) + \beta(B)/3$, hence $\beta(B) \leq 3\beta(\{x_k : k \geq 1\})$. □

In the situation described by Theorem 3.4, the set of all mild solutions of (1) is a compact subset of $C(J;X)$ for every $u_0 \in \overline{D(A)}$; recall that all fixed points of G are in the compact set K. Consequently, the subsequent example shows that the method used above does not work, without additional assumptions, in general Banach spaces. In this example we define an m-accretive operator A in an appropriate Banach space X such that $-A$ generates an equicontinuous semigroup and the set of all mild solutions of

$$u' \in -Au + C \quad \text{on } [0,1], \qquad u(0) = 0 \tag{16}$$

is not relatively compact, although $C \subset X$ is compact. By means of this example it is also clear that Lemma 3.7 is false without the extra assumption on X^*.

Example 3.2 Let $X = \{u \in C_b(\mathbb{R}_+) : u(0) = 0\}$ with the sup-norm $|\cdot|_0$. For $u \in X$ let u^+ be defined by $u^+(x) = \max_{[0,x]} u(s)$, i.e. u^+ is the smallest increasing function such that $u^+(x) \geq u(x)$ on $\mathbb{R}_+$. Now define $A : D \to 2^X \setminus \emptyset$ by means of

$$Au = \{v - u : v \in X, v^+ = u\} \quad \text{on } D = \{u^+ : u \in X\}.$$

1. We claim that A is m-accretive such that $-A$ generates an equicontinuous semigroup. For this purpose let us first prove that

$$(u^+ + \alpha(u - u^+))^+ = u^+ \qquad \text{for all } u \in X \text{ and } \alpha > 0. \tag{17}$$

Let $v = u^+ + \alpha(u - u^+)$. Then $u \leq u^+$ implies $v \leq u^+$ hence also $v^+ \leq u^+$. On the other hand, given $x \geq 0$ there is $\tau \in [0, x]$ with $u^+(x) = u(\tau)$ which yields $u^+(s) = u(\tau)$ for all $s \in [\tau, x]$. This implies $v^+(x) \geq v(\tau) = u^+(\tau) + \alpha(u(\tau) - u^+(\tau)) = u^+(x)$, hence (17) holds. To show $R(I + \lambda A) = X$ for all $\lambda > 0$, let $w \in X$ be given and $v := w^+ + \frac{w-w^+}{\lambda}$. Then $u := v^+ \in D$ and $v^+ = w^+ = u$ by (17), hence $\frac{w-w^+}{\lambda} \in Au$ which means $w \in u + \lambda Au$. Moreover, $u = w^+$ is the only solution of $w \in u + \lambda Au$, since $w \in \hat{u} + \lambda A\hat{u}$ with $\hat{u} \in D$ implies $w = \hat{u} + \lambda(\hat{v} - \hat{u})$ for some $\hat{v} \in X$ with $\hat{v}^+ = \hat{u}$, hence $\hat{u} = w^+$ by (17). Therefore, given $\lambda > 0$, $J_\lambda = (I + \lambda A)^{-1} : X \to D$ is well defined and given by $J_\lambda w = w^+$ on X. It remains to show that A is accretive which follows if all J_λ are nonexpansive maps, i.e. $|u^+ - v^+|_0 \leq |u - v|_0$ for all $u, v \in X$. Suppose, on the contrary, that $|u - v|_0 < |u^+(x) - v^+(x)|$ for some $x > 0$, where we may assume $u^+(x) > v^+(x)$. Since $u^+(x) = u(\tau)$ for some $\tau \in [0, x]$ and $v^+(x) \geq v^+(\tau) \geq v(\tau)$, this gives the contradiction $|u(\tau) - v(\tau)| < u^+(x) - v^+(x) \leq u(\tau) - v(\tau)$. Evidently, $J_\lambda u = u^+$ on X for all $\lambda > 0$ and $u = u^+$ on $\overline{D} = D$ imply

$$S(t)u = \lim_{n\to\infty} J^n_{t/n} u = u \qquad \text{for all } t \geq 0 \text{ and } u \in D.$$

Hence the semigroup generated by $-A$ is given by $S(t) = I_{|D}$ for all $t \geq 0$, which is obviously equicontinuous.

2. Let $J = [0, 1]$, $J_{n,l} = [\frac{l}{n}, \frac{l+1}{n})$ for $n \geq 1$, $l = 0, \ldots, n-1$ and $w_n \in L^1(J; X)$ given by $w_n(t) = (-1)^l \varphi$ on $J_{n,l}$, where $\varphi \in X$ is the sawtooth-function defined by $\varphi(x) = \int_0^x \psi(s)\, ds$ on $\mathbb{R}_+$ with $\psi = \sum_{l\geq 0} (-1)^l \chi_{[2l-1,2l+1)}$. Evidently, $w_n(t) \in C := \{-\varphi, \varphi\}$ on J for all $n \geq 1$. Hence $(w_n) \subset L^1(J; X)$ is weakly relatively compact and also $w_n \rightharpoonup 0$. Let $u_n = \mathcal{S}w_n$ be the mild solution of $u' + Au \ni w_n(t)$ on J, $u(0) = 0$. We claim that (u_n) is not relatively compact in $C(J; X)$. For a proof, it suffices to show that $(u_{2j}(1)) \subset X$ is not relatively compact. We will only sketch the proof, since the details require lengthy but elementary calculations. Fix an even $n \geq 2$ and let $t_k = \frac{2k}{n}$ for $k \geq 0$. Then by induction w.r. to k one can show

$$u(t_k)(x) = \frac{1}{n} \int_0^x (-\psi)^+(s) \chi_{[0,2k+1]}(s)\, ds \qquad \text{on } \mathbb{R}_+, \text{ for every } k \geq 0. \tag{18}$$

To get this representation the first step is to solve the initial value problem

$$v' \in -Av + \varphi \qquad \text{on } [0,1/n], \;\; v(0) = u(t_k).$$

Since the operator $\tilde{A}$, defined by $\tilde{A}u = Au - \varphi$, is again m-accretive the integral solution of this problem is given by the exponential formula, namely $v(t) = \lim_{m\to\infty} \tilde{J}^m_{t/m} u(t_k)$ where $\tilde{J}_\lambda u = J_\lambda(u+\lambda\varphi) = (u+\lambda\varphi)^+$. Using representation (18) it turns out that $v(t) = (u(t_k)+t\varphi)^+$ for $t \in [0,1/n]$, hence $u(t_k+1/n) = v(1/n) = (u(t_k)+\varphi/n)^+$. Then the next step is to take this as the new initial value and to solve $v' \in -Av - \varphi$ on $[1/n, 2/n]$. Evidently $u(t_{k+1}) = v(2/n)$ and a similar argument as above yields

$$u(t_{k+1}) = ((u(t_k) + \frac{1}{n}\varphi)^+ - \frac{1}{n}\varphi)^+.$$

Finally, it can be checked by elementary calculations that $u(t_{k+1})$ is again of the type given by (18). Now it is easy to conclude that u_n, for even n, satisfies

$$|u_n(1)(x) - \frac{x}{2n}| \leq \frac{1}{2n} \;\text{ on } [0,2n], \quad u_n(1)(x) = 1 \;\text{ on } [2n,\infty).$$

Therefore, $u_{2j}(1)(x) \to 0$ as $j \to \infty$ uniformly on bounded intervals, but $|u_{2j}(1)|_0 = 1$, hence $(u_{2j}(1))$ is not relatively compact which proves the claim. $\diamond$

The main ingredients in Example 3.2 are taken from a counter-example due to M. Pierre which can be found in Vrabie [112] on p.224ff; there it is shown that a certain sequence of approximate solutions for (16) is not relatively compact.

While Examples 3.1 and 3.2 show that, in general, assumptions on X (resp. on X^*) are needed, there are two relevant special cases where the fixed point approach works within general Banach spaces: existence results are valid if A is linear, densely defined and m-accretive, or if $A : X \to X$ is continuous and accretive. In fact, it is possible to combine these results, i.e. to allow for "semilinear" operators A of the type

$$Au = A_0 u + g(u) \;\text{ on } D(A) := D(A_0), \tag{19}$$

where $A_0 : D(A_0) \to X$ is linear, m-accretive with $\overline{D(A_0)} = X$ and $g : X \to X$ is continuous, accretive. In this case A itself is also m-accretive (see Theorem 5.5), hence (2) has a unique mild solution $\mathcal{S}w$ for every $w \in L^1(J;X)$ and $u_0 \in X$. On the other hand, Theorem 3.5 applies to

$$u' + A_0 u = w(t) - g(u) \;\text{ on } J, \;\; u(0) = u_0. \tag{20}$$

If u denotes the corresponding mild solution of (20), then it is easy to verify that u is also an integral solution of (2), hence $u = \mathcal{S}w$. By means of this simple observation, it is possible to exploit the semilinear structure of A, since Proposition 1.5 implies that $u = \mathcal{S}w$ satisfies

$$u(t) = S_0(t)u_0 + \int_0^t S_0(t-s)(w(s) - g(u(s)))ds \;\text{ on } J, \tag{21}$$

where $S_0(t)$ denotes the C_0-semigroup generated by $-A_0$. Actually, in this situation $u \in C(J;X)$ is a mild solution of (2) iff u satisfies (21). Now we have

Theorem 3.5 *Let X be a real Banach space and A be given by (19), where $A_0 : D(A_0) \to X$ is linear, m-accretive with $\overline{D(A_0)} = X$ and $g : X \to X$ is continuous, accretive. Let $J = [0,a] \subset \mathbb{R}$ and $F : J \times X \to 2^X \setminus \emptyset$ with closed convex values satisfying (3) and (13) be such that $F(\cdot, x)$ has a strongly measurable selection for every x and $F(t,\cdot)$ is weakly usc for almost all $t \in J$. Then (1) has a mild solution for every $u_0 \in X$.*

Proof. 1. In the situation under consideration, the following variant of Lemma 3.7 holds. Let $(w_k) \subset L^1(J;X)$ satisfy $|w_k(t)| \le \varphi(t)$ a.e. on J with $\varphi \in L^1(J)$ and let X_0 be a closed separable subspace of X such that $w_k(t) \in X_0$ a.e. on J. Then

$$\beta(\{(\mathcal{S}w_k)(t) : k \ge 1\}) \le \int_0^t \beta_{X_0}(\{w_k(s) : k \ge 1\})ds \;\text{ on } J; \tag{22}$$

notice that such a subspace X_0 always exists since the w_k are strongly measurable. Inspection of the proof of Lemma 3.7 shows that the same arguments apply if we replace Y by X_0 there, given that we are able to show $\beta(\Omega_n^\epsilon) = 0$. Actually, we will show the following more general fact. Let $(w_k) \subset L^1(J;X)$ satisfy $|w_k(t)| \le \varphi(t)$ a.e. on J with $\varphi \in L^1(J)$ as well as $w_k(t) \in C(t)$ a.e. on J with compact sets $C(t) \subset X$. Then $\{(\mathcal{S}w_k)(t) : k \ge 1\}$ is relatively compact for all $t \in J$.
Let (w_k) be such a sequence, where we may assume $w_k \rightharpoonup w$ in $L^1(J;X)$ due to Lemma 3.2. We claim that $\mathcal{S}w_k \to \mathcal{S}w$ in $C(J;X)$. By the remarks in front of this theorem, $u_k = \mathcal{S}w_k$ and $u = \mathcal{S}w$ satisfy

$$u_k(t) = S_0(t)u_0 + \int_0^t S_0(t-s)(w_k(s) - g(u_k(s)))ds \;\text{ on } J,$$

respectively

$$u(t) = S_0(t)u_0 + \int_0^t S_0(t-s)(w(s) - g(u(s)))ds \;\text{ on } J.$$

Consider first the "harmless" parts

$$z_k(t) = \int_0^t S_0(t-s)w_k(s)ds \quad\text{and}\quad z(t) = \int_0^t S_0(t-s)w(s)ds \;\text{ on } J.$$

Then $z_k \to z$ in $C(J;X)$, which can be seen as follows. Since $S_0(t-\cdot)w_k(\cdot)$ is strongly measurable, the operators $S_0(t)$ are nonexpansive and $\beta(\{w_k(s) : k \ge 1\}) \le \beta(C(s)) = 0$ a.e. on J, application of Proposition 3.1 (with an appropriate separable subspace X_0 instead of X) shows that the sections $\{z_k(t) : k \ge 1\}$ are relatively compact. Given $0 \le s \le t, \bar{t} \le a$, the inequality for integral solutions implies

$$\begin{aligned} |z_k(t) - z_k(\bar{t})| &\le |S_0(t-s)z_k(s) - S_0(\bar{t}-s)z_k(s)| + \int_s^t \varphi(\tau)d\tau + \int_s^{\bar{t}} \varphi(\tau)d\tau \\ &\le |S_0(|t-\bar{t}|)z_k(s) - z_k(s)| + \int_s^t \varphi(\tau)d\tau + \int_s^{\bar{t}} \varphi(\tau)d\tau. \end{aligned}$$

Now if (z_k) is not equicontinuous, then $|z_k(t_k) - z_k(\bar{t}_k)| \geq \epsilon_0 > 0$ with $t_k \to t$, $\bar{t}_k \to t$ and $t = 0$ is not possible. Since $(z_k(s))$ is relatively compact for every $s \in J$, the estimate above yields the contradiction $\epsilon_0 \leq 2\int_s^t \varphi(\tau)d\tau$ for all $s \in [0,t)$. Therefore (z_k) is relatively compact in $C(J;X)$. Let (z_{k_l}) be a convergent subsequence of (z_k). Then its limit is z, since $x^*(z_{k_l}(t)) \to x^*(z(t))$ for every $x^* \in X^*$ and $t \in J$. Hence z is the only accumulation point of (z_k) and therefore $z_k \to z$.

Now consider

$$v_k(t) = -\int_0^t S_0(t-s)g(u_k(s))ds \quad \text{and} \quad v(t) = -\int_0^t S_0(t-s)g(u(s))ds \quad \text{on } J.$$

Evidently $v_k - v = u_k - u - (z_k - z)$, v is the mild solution of

$$v' + A_0 v = -g(u(t)) \;\; \text{on } J, \;\; v(0) = 0,$$

and v_k is the mild solution of the same initial value problem with $g(u_k(t))$ instead of $g(u(t))$. Exploitation of the inequality for integral solutions and the fact that g is also s-accretive yields

$$\begin{aligned}|v_k(t) - v(t)| &\leq \int_0^t [v_k(s) - v(s), -g(u_k(s)) + g(u(s))]ds \\ &\leq \int_0^t [u_k(s) - (u(s) + e_k(s)), -g(u(s) + e_k(s)) + g(u(s))]ds \\ &\leq \int_0^t |g(u(s) + e_k(s)) - g(u(s))|ds \quad \text{with } e_k := z_k - z.\end{aligned}$$

Hence $|v_k - v|_0 \to 0$ by the dominated convergence theorem and therefore

$$|u_k - u|_0 \leq |v_k - v|_0 + |z_k - z|_0 \to 0,$$

i.e. $\mathcal{S}w_k \to \mathcal{S}w$ in $C(J;X)$ as claimed above.

2. As before, we consider $G = \mathcal{S} \circ \mathrm{Sel}$ and get a closed bounded convex $K_0 \subset C(J;X)$ such that $G(K_0) \subset K_0$ by (3). Let $K_{n+1} = \overline{\mathrm{conv}} G(K_n)$ and $\rho_n(t) = \beta(K_n(t))$ on J for all $n \geq 0$. Obviously $\rho_n(t) \searrow \rho_\infty(t) \geq 0$ for every $t \in J$ with some $\rho_\infty(\cdot)$, and we claim that $\rho_\infty(t) = 0$ on J. Now, since measurability of ρ_n is not clear, let $r_n : J \to \mathbb{R}_+$ be measurable with $r_n \leq \rho_n$ such that $r \leq \rho_n$ with a measurable $r : J \to \mathbb{R}_+$ implies $r \leq r_n$ a.e. on J. To obtain such functions r_n, fix $n \geq 0$ and notice that ρ_n is bounded since $K_0 \subset B_R(0)$, say. Hence there are $\psi_k \in L^1(J)$ with $\psi_k \leq \rho_n$ such that

$$\int_J \psi_k(t)dt \to \sup\{\int_J \psi(t)dt : \psi \in L^1(J), \psi \leq \rho_n\}.$$

The ψ_k can be chosen such that $\psi_k \leq \psi_{k+1}$ and then $r_n = \sup_{k\geq 1} \psi_k$ does the job.
We will show that

$$\rho_{n+1}(t) \leq 12\int_0^t k(s)r_n(s)ds \;\; \text{on } J \text{ for all } n \geq 0. \tag{23}$$

Fix $t \in J$, where it suffices to consider $t > 0$. Then $\rho_{n+1}(t) = \beta(G(K_n)(t))$ and by the last step in the proof of Theorem 3.4 there is $(x_k) \subset G(K_n)(t)$ such that $\rho_{n+1}(t) \leq 3\beta(\{x_k : k \geq 1\})$. Of course $x_k = (\mathcal{S}w_k)(t)$ with $w_k \in \mathrm{Sel}\,(u_k)$ for certain $u_k \in K_n$. Since $|w_k(s)| \leq c(s)(1+R)$ a.e. on J and $X_0 = \overline{\mathrm{span}}\Big(\bigcup\limits_{k\geq 1} w_k(J) \cup \bigcup\limits_{k\geq 1} u_k(J)\Big)$ is separable (eventually after a change of the w_k on a null set), inequality (22) applies and yields

$$\rho_{n+1}(t) \leq 3\beta(\{x_k : k \geq 1\}) \leq 3\int_0^t \beta_{X_0}(\{w_k(s) : k \geq 1\})ds.$$

Exploitation of (13) and $\beta_{X_0}(B) \leq 2\beta(B)$ for bounded $B \subset X_0$ implies

$$\beta_{X_0}(\{w_k(s) : k \geq 1\}) \leq 2k(s)\beta(\{u_k(s) : k \geq 1\}) \leq 2k(s)\beta_{X_0}(\{u_k(s) : k \geq 1\}),$$

and $\gamma(\cdot) := \beta_{X_0}(\{u_k(\cdot) : k \geq 1\})$ is measurable due to the representation of β_{X_0} given in Proposition 3.2. Moreover $\gamma(t) \leq 2\beta(K_n(t)) = 2\rho_n(t)$ on J, hence $\gamma(t)/2 \leq r_n(t)$ a.e. on J and therefore (23) holds. Consequently, application of Fatou's lemma shows that

$$0 \leq r_\infty(t) \leq 12\int_0^t k(s)r_\infty(s)ds \ \text{ on } J$$

for $r_\infty(\cdot) = \overline{\lim\limits_{n\to\infty}}\, r_n(\cdot)$. Since the same inequality holds for $r(t) = \int_0^t k(s)r_\infty(s)ds$ instead of r_∞, Gronwall's lemma implies $r = r_\infty = 0$. Then $\rho_\infty(t) = 0$ on J is a consequence of (23).

The arguments given so far imply $\beta(K(t)) \equiv 0$ where $K = \bigcap\limits_{n\geq 0} K_n$. To show $K \neq \emptyset$, pick $u_n \in G(K_n) \subset K_{n+1}$ for every $n \geq 0$. Then $\beta(\{u_n(t) : n \geq 1\} = \beta(\{u_n(t) : n \geq m\}) \leq \rho_m(t)$ on J for all $m \geq 1$, hence $(u_n) \subset G(K_0)$ has relatively compact sections. Evidently $u_n = \mathcal{S}w_n$ for certain w_n such that $|w_n(t)| \leq \varphi(t) := c(t)(1+R)$ a.e. on J, hence the inequality for integral solutions implies

$$|u_n(t) - u_n(\bar{t})| \leq |S(|t-\bar{t}|)u_n(s) - u_n(s)| + \int_s^t \varphi(\tau)d\tau + \int_s^{\bar{t}} \varphi(\tau)d\tau$$

for all $0 \leq s \leq t, \bar{t} \leq a$. This yields equicontinuity of (u_n). Hence $u_{n_k} \to u$ in $C(J;X)$ for some subsequence, and then $u \in K$. The same arguments show that $G(K)$ is equicontinuous, hence relatively compact. Therefore, $\hat{K} = \overline{\mathrm{conv}}G(K)$ is nonempty and compact convex with $G(\hat{K}) \subset \hat{K}$.

It remains to show that G has closed graph. If this holds then Lemma 2.1 yields a fixed point of G, i.e. a mild solution of (1); recall that $G(u)$ is contractible by step 3 of the proof to Lemma 3.3. Let $v_n \in G(u_n)$ with $u_n \to u$ and $v_n \to v$ in $C(J;X)$. Then $v_n = \mathcal{S}w_n$ with $w_n \in \mathrm{Sel}\,(u_n)$, hence $|w_n(t)| \leq \varphi(t)$ a.e. on J with $\varphi \in L^1(J)$ by (3), and (13) implies $\beta(\{w_n(t) : n \geq 1\}) = 0$ a.e. on J. We may then assume $w_n \rightharpoonup w$ in $L^1(J;X)$, and in this situation $\mathcal{S}w_n \to \mathcal{S}w$ in $C(J;X)$ has been shown in step 1 of this proof. We are done since

Sel is weakly usc with weakly compact convex values by step 2 of the proof of Lemma 3.3 and therefore $w \in \mathrm{Sel}\,(u)$, i.e. $v = \mathcal{S}w \in G(u)$. $\square$

In Theorem 3.5, time-dependence of g has been excluded only in order to remain within the framework of m-accretive operators. Let us briefly explain how an extension to the time-dependent case

$$A(t)u = A_0 u + g(t,u), \quad t \in J,\ u \in D(A(t)) := D(A_0) \tag{24}$$

with A_0 as in Theorem 3.5 and Carathéodory $g : J \times X \to X$ can be achieved. If the $g(t,\cdot)$ are accretive then $A(t)$ is m-accretive for every $t \in J$, but this situation is not covered by the existing theory for time-dependent m-accretive operators where further assumptions concerning the t-dependence are needed (see e.g. Pavel [92]) to obtain mild solutions of

$$u' + A(t)u = w(t) \ \text{ on } J,\ \ u(0) = u_0 \tag{25}$$

for $w \in L^1(J;X)$. Therefore it is favorable to exploit the semilinear structure as explained in front of Theorem 3.5: if $A(t)$ is of the type given above, then $u \in C(J;X)$ is said to be a mild solution of (25) if

$$u(t) = S_0(t)u_0 + \int_0^t S_0(t-s)(w(s) - g(s,u(s)))ds \ \text{ on } J,$$

where $S_0(t)$ is the semigroup generated by $-A_0$. By Theorem 3.2 and Proposition 1.5 it is then clear that (25) has a unique mild solution $u =: \mathcal{S}w$ for every $w \in L^1(J;X)$, if g also satisfies a growth condition of type (6). Moreover, if u and $\overline{u}$ are mild solutions of (25) corresponding to w and $\overline{w}$, respectively, it is easy to check that

$$|u(t) - \overline{u}(t)| \le |u(s) - \overline{u}(s)| + \int_s^t |w(\tau) - \overline{w}(\tau)|d\tau \ \text{ for } 0 \le s \le t \le a$$

holds again. Now an inspection of the proof of Theorem 3.5 shows that all arguments (with obvious modifications) also apply in this time-dependent setting. We therefore have

Theorem 3.6 *Let X be a real Banach space, $J = [0,a] \subset \mathbb{R}$ and $A(t)$ be given by (24), where $A_0 : D(A_0) \to X$ is linear, m-accretive with $\overline{D(A_0)} = X$ and $g : J \times X \to X$ is Carathéodory such that $g(t,\cdot)$ is accretive for all $t \in J$ and $|g(t,x)| \le d(t)(1+|x|)$ on $J \times X$ with $d \in L^1(J)$. Let $F : J \times X \to 2^X \setminus \emptyset$ with closed convex values satisfying (3) and (13) be such that $F(\cdot,x)$ has a strongly measurable selection for every x and $F(t,\cdot)$ is weakly usc for almost all $t \in J$. Then (1) has a mild solution for every $u_0 \in X$.*

3.5 Remarks

Remark 3.1 Theorems 3.1-3.4 remain valid if A is m-ω-accretive for some $\omega \in \mathbb{R}$, i.e. if $A_\omega := A + \omega I$ is m-accretive, since the corresponding result applies to A_ω and $F_\omega := F + \omega I$

(resp. $f_\omega := f + \omega I$) instead of A and F (resp. f) in each of these cases. Notice in particular that if $-A$ generates an equicontinuous or compact semigroup then this property is inherited to the semigroup generated by $-A_\omega$, which can be checked easily by means of the inequality for integral solutions and the fact that $(I + \lambda A_\omega)^{-1} = (I + \frac{\lambda}{1+\lambda\omega} A)^{-1} \circ \frac{1}{1+\lambda\omega} I$. This leads to a mild solution of the original problem, since u is a mild solution of $u' + A_\omega u \ni w(t) + \omega u$ for $w \in L^1(J; X)$ iff u is a mild solution of $u' + Au \ni w(t)$.
Of course similar modifications of Theorems 3.5 and 3.6 can be obtained as well. For instance, Theorem 3.6 remains valid if $g(t, \cdot)$ is $\omega(t)$-accretive with $\omega \in L^1(J)$, i.e. if $-g$ satisfies a dissipativity condition of type (5).

Remark 3.2 Theorem 3.1 is a compilation of essentially known results that are mentioned below; the unified proof by means of Lemma 3.3 is based on Bothe [23]. A local version of Theorem 1 can be obtained as follows: Let F be defined on $J \times D_r$ with $D_r = B_r(x_0) \cap \overline{D(A)}$ and suppose that the corresponding assumptions of Theorem 3.1 hold. Then Theorem 3.1 applies to $\tilde{F}$ given by $\tilde{F}(t, x) = F(t, P(R(x)))$, where P is as in step 1 of the proof of Lemma 3.3 and R is the radial retraction onto $\overline{B}_\delta(x_0)$ with $\delta > 0$ such that $P(\overline{B}_\delta(x_0)) \subset B_r(x_0)$. This yields a mild solution of (1) with $\tilde{F}$ which is a local mild solution of (1) with F, since $F(t, x) = \tilde{F}(t, x)$ on $J \times (B_\delta(x_0) \cap \overline{D(A)})$. Such a local version of part (b) of Theorem 3.1 comes close to Theorem 3.3.1 in Vrabie [112]; there it is assumed (in difference to the conditions imposed in Theorem 3.1(b)) that X is separable and the $F(\cdot, x)$ are measurable.

Theorem 3.1(b) includes Theorem 2.1 in Tolstonogov/Umanskii [105], where the $F(t, \cdot)$ are assumed to be ϵ-δ-usc, and part (c) contains Theorem 2.2 of the same paper; there the differing assumptions are separability of X and measurability of $F(\cdot, u(\cdot))$ for every $u \in C(J; \overline{D(A)})$.
Let us note that Theorem 3.1(d) remains valid if A is of the type $A = A_0 + g$, where A_0 is linear, m-accretive, densely defined and $g : X \to X$ is continuous, accretive. This follows by reduction to $g = 0$: since all fixed points of $G = \mathcal{S} \circ \mathrm{Sel}$ are contained in the compact set K given in the proof of Lemma 3.3, we may assume the g is bounded. Then Theorem 3.1(d) applies to A_0 and $F_0 := F - g$ instead of A and F, and the corresponding solution is a solution of the original problem as well. This extension of part (d) contains Theorem 2.3 in Tolstonogov/Umanskii [105] where again separability of X and measurability of $F(\cdot, u(\cdot))$ for every $u \in C(J; \overline{D(A)})$ is assumed.

Under fairly restrictive assumptions one can exploit additional knowledge about the regularity of mild solutions of (2) to obtain a strong solution of (1). To be more specific, consider the situation described by Theorem 3.1(b) in case X is a Hilbert space, and let u be a mild solution of (1). Then Proposition 3.8 in Brezis [29] says that u is a strong solution if $\dim X < \infty$, while the same conclusion follows from Theorem 3.6 in Brezis [29] if $A = \partial\varphi$ with a proper lsc convex function φ and $c \in L^2(J)$ in (3). The corresponding existence results for (1) have been established in parts III and IV of Attouch/Damlamian [6].

Remark 3.3 If Theorem 3.3 is specialized to the case $A = 0$, the resulting solution of (1) is obviously a strong solution. Therefore Theorem 3.3 extends Theorem 10.5 in Deimling [42], where X is a Hilbert space and the $F(t,\cdot)$ are assumed to be usc; parts of the proof of Theorem 3.3 are taken from this reference.

Remark 3.4 Lemma 3.7 and Theorem 3.4 are taken from Bothe [23]. If F is only defined on $J \times D_r$ with $D_r = B_r(x_0) \cap \overline{\text{conv}} D(A)$, a corresponding local version follows immediately, since (3) on $J \times D_r$ implies $|u(t) - u_0| < r$ on $[0, b)$ for every mild solution u of (1) if $b > 0$ is sufficiently small. Notice that the $F(t,\cdot)$ have to be defined on a convex set to obtain the convex (compact) K, and here we cannot use the map P from the proof of Lemma 3.3 to extend F to all of $J \times X$, since it is unclear whether this extension will satisfy (13). On the other hand, in applications where X^* is uniformly convex, X will usually also have this property and in this case $\overline{D(A)}$ is convex. Apart from this detail, such a local version of Theorem 3.4 is a considerable improvement of Theorem 3.6.1 in Vrabie [112], where local mild solutions are obtained in the following situation: X separable with X^* uniformly convex, A m-accretive such that $-A$ generates an equicontinuous semigroup, $F : [a, b] \times V \to 2^X \setminus \emptyset$ jointly usc with closed convex values satisfying $\beta(F([a, b] \times B)) = 0$ for all bounded $B \subset V$, where V is open in $\overline{D(A)}$ with $u_0 \in V$.
Specialized to the case of a single-valued compact perturbation, Theorem 3.4 (in its local formulation) becomes essentially Theorem 3.3 in Gutman [64].

Remark 3.5 Theorems 3.5 and 3.6 are new; the latter is a combination of Theorems 3 and 4 in Bothe [23]. The usual local version of Theorem 3.5 extends Theorem 2 in Schechter [100], where single-valued compact perturbations have been considered.

If Theorem 3.6 is specialized to $A_0 = 0$ and single-valued $F = \{f\}$, it includes the main result (Satz 2.3) in §2 of Schmidt [101]. There it is assumed that $f, g : J \times X \to X$ are continuous and bounded such that $\beta(f(J \times B)) \le L\beta(B)$ for all bounded $B \subset X$ and $g(t,\cdot)$ is ω-accretive. The latter is an extension of Theorem 2 in Volkmann [108], which is the local version for compact f. Remember Remark 3.1 and observe that mild solutions are even continuously differentiable in this case.

Remark 3.6 Let us add some information concerning problem (1) in case of lower semicontinuous perturbations. If F is lsc with closed convex values the problem can be reduced to the single-valued continuous case by application of Michael's selection theorem. In the more interesting case when the values of F are only closed and bounded, this simple reduction is not possible but selection results are still helpful. For instance if $F : J \times \overline{D(A)} \to 2^X \setminus \emptyset$ is lsc with closed bounded values such that (3) is satisfied, one may use the fixed point approach. Due to the considerations in §3.2 it suffices to consider $G = \mathcal{S} \circ \text{Sel}$ on a compact convex set $K \subset C(J; X)$ if the semigroup generated by $-A$ is compact, say. In this situation it is possible to prove that $\mathrm{Sel} : K \to 2^{L^1(J;X)} \setminus \emptyset$ is lsc, and the values $\mathrm{Sel}\,(u)$ need not be convex but are decomposable, i.e. $w_1, w_2 \in \mathrm{Sel}\,(u)$ implies $\chi_A w_1 + (1-\chi_A) w_2 \in \mathrm{Sel}\,(u)$ for every measurable $A \subset J$. Due to these properties of $\mathrm{Sel}\,(\cdot)$, the selection theorem in Fryszkowski [57] yields a continuous selection $h : K \to L^1(J;X)$ of this map, hence (1) has a mild solution by Schauder's fixed point theorem. The corresponding existence result is essentially Theorem 3.1 in Mitidieri/Vrabie [84]. Existence of strong solutions in the finite dimensional case $X = (\mathbb{R}^n, |\cdot|_2)$ was obtained before in Colombo/Fonda/Ornelas [36] by means of similar arguments for jointly measurable F such that F is lsc in x.

§4 Invariance and Viability

In the present section we concentrate on evolution problems with single-valued perturbations. Let A be an m-accretive operator in a real Banach space X, $J = [0,a] \subset \mathbb{R}$ and $f : J \times \overline{D(A)} \to X$. Given a "tube" $K : J \to 2^X$ with closed values $K(t)$ such that $K_A(t) := K(t) \cap \overline{D(A)} \neq \emptyset$ on J, we look for a mild solution u of

$$u' + Au \ni f(t,u) \quad \text{on } J, \quad u(0) = u_0 \tag{1}$$

that satisfies the time-dependent constraints $u(t) \in K_A(t)$ on J; if this holds u is said to be a viable solution. Now notice that any reasonable sufficient condition will imply that, given $t_0 \in [0,a)$ and $u_0 \in K_A(t_0)$, the initial value problem

$$u' + Au \ni f(t,u) \quad \text{on } [t_0,a], \quad u(t_0) = u_0 \tag{2}$$

has a mild solution such that $u(t) \in K_A(t)$ on $[t_0,a]$. In this situation the tube $K(\cdot)$ is called weakly positively invariant or, alternatively, $K(\cdot)$ is said to have the viability property (for $u' + Au \ni f(t,u)$); here "weakly" refers to the fact that other solutions may leave the tube. We say that $K(\cdot)$ is positively invariant (for $u' + Au \ni f(t,u)$), if all solutions starting in $\mathrm{gr}\,(K_A)$ remain in this set.

In case we are interested in existence of a viable solution of (1), it is of course possible to incorporate the constraints into the evolution problem, simply by considering f to be defined on $\mathrm{gr}\,(K_A)$ only.

4.1 Approximate solutions

Consider initial value problem (1) in the situation described above with continuous right-hand side $f : \mathrm{gr}\,(K_A) \to X$. Then a necessary condition for weak positive invariance of $K(\cdot)$ can be obtained as follows. Suppose that (2) has mild solution u and let v be the mild solution of

$$v' + Av \ni f(t_0,u_0) \quad \text{on } [t_0,a], \quad v(t_0) = u_0.$$

By continuity of f and u it follows that

$$\frac{1}{h}|u(t_0+h) - v(t_0+h)| \leq \frac{1}{h}\int_{t_0}^{t_0+h} |f(t,u(t)) - f(t_0,u_0)|dt \to 0 \quad \text{as } h \to 0+,$$

hence

$$\varliminf_{h\to 0+} h^{-1}\rho(S_{f(t_0,u_0)}(h)u_0, K_A(t_0+h)) = 0,$$

where $S_z(\cdot)$ denotes the semigroup generated by $-A_z$ with $A_z x := Ax - z$ on $D(A_z) = D(A)$. Consequently,

$$f(t,x) \in T_K^A(t,x) \quad \text{for all } (t,x) \in \mathrm{gr}(K_A) \text{ with } t < a \tag{3}$$

is a necessary condition for weak positive invariance of $K(\cdot)$, where T_K^A is defined on $\mathrm{gr}(K_A)\cap([0,a)\times X)$ by

$$T_K^A(t,x)=\{z\in X:\ \varliminf_{h\to 0+} h^{-1}\rho(S_z(h)x,K_A(t+h))=0\}.$$

In the special case $A=0$ this becomes

$$T_K(t,x)=\{z\in X:\ \varliminf_{h\to 0+} h^{-1}\rho(x+hz,K(t+h))=0\},$$

and if, in addition, $K(t)\equiv K$ holds then $T_K(t,x)=T_K(x)$ is the Bouligand contingent cone with respect to K at the point x introduced in §2.3.

Since all $K(t)$ are closed by assumption, it is also natural to assume that $\mathrm{gr}(K_A)$ is closed from the left, i.e.

$$(t_n)\subset J \text{ with } t_n\nearrow t \text{ and } x_n\in K_A(t_n) \text{ with } x_n\to x \ \text{ implies } \ x\in K_A(t);$$

notice that if there are mild solutions u_n with $u_n(t_n)=x_n$, then $K_A(t)\ni u_n(t)\to x$.

In the subsequent sections we will show that the "subtangential condition" (3) is also sufficient for existence of a (viable) solution in several situations. The next result is a basic step in this direction, since it provides appropriate approximate solutions.

In the sequel $u(\cdot\,;t_0,u_0,w)$ denotes the mild solution of

$$u'+Au\ni w(t) \ \text{ on } [t_0,a],\ \ u(t_0)=u_0, \tag{4}$$

and if $w\in L^1(J;X)$ then $u(\cdot\,;t_0,u_0,w)$ is short for $u(\cdot\,;t_0,u_0,w_{|[t_0,a]})$. With this notations, the semigroup property of solutions reads

$$u(t;\tau,u_0,w)=u(t;\overline{\tau},u(\overline{\tau};\tau,u_0,w),w) \quad \text{for all } 0\le\tau\le\overline{\tau}\le t\le a.$$

If $t_0=0$ and u_0 is fixed we simply write $u(\cdot\,;w)$ instead of $u(\cdot\,;t_0,u_0,w)$.

Lemma 4.1 *Let A be m-accretive in a real Banach space X, $J=[0,a]\subset\mathbb{R}$ and $K:J\to 2^X$ be such that $K_A(0)\neq\emptyset$ and $gr(K_A)$ is closed from the left. Let $f:gr(K_A)\to X$ be bounded and such that (3) is satisfied. Then, given $u_0\in K_A(0)$ and $\epsilon>0$, there is $w\in L^1(J;X)$ such that*

$$w(t)\in f([J_{t,\epsilon}\times\overline{B}_{\gamma\epsilon}(u(t;w))]\cap gr(K_A)) \quad \text{a.e. on } J \tag{5}$$

with $\gamma=1+a$, where $J_{t,\epsilon}=[t-\epsilon,t]\cap J$.

Proof. Let $u_0\in K_A(0)$ and $\epsilon>0$, where we may assume $\epsilon\le 1$. Consider the set M^ϵ of approximate solutions defined by

$$\begin{aligned} M^\epsilon \ &=\{(v,w,P,b):b\in(0,a],\\ &\quad v:[0,b]\to X \text{ with } v(b)\in K_A(b), v([0,b]) \text{ relatively compact},\\ &\quad w:[0,b]\to X \text{ strongly measurable such that (5) holds a.e. on } [0,b],\\ &\quad P\subset[0,b) \text{ with } 0\in P, b\in\overline{P} \text{ such that } \tau\in P \text{ implies } v(\tau)\in K_A(\tau)\\ &\quad \text{and } |v(t)-u(t;\tau,v(\tau),w)|\le\epsilon(t-\tau) \text{ on } [\tau,b]\}, \end{aligned}$$

and notice that we are done if M^ϵ contains an element with $b = a$.

1. We claim that M^ϵ is nonempty. Since $z_0 := f(0, u_0) \in T_K^A(0, u_0)$, there is $h \in (0, \epsilon]$ such that $y_1 := S_{z_0}(h)u_0$ satisfies $\rho(y_1, K_A(h)) \leq \frac{1}{2}\epsilon h$, hence there is $u_1 \in K_A(h)$ such that $|e_0| \leq \epsilon$ for $e_0 := \frac{u_1 - y_1}{h}$. Let $t_0 = 0$, $t_1 = t_0 + h$ and

$$v(t) = S_{z_0}(t - t_0)u_0 + (t - t_0)e_0 \quad \text{on } [t_0, t_1].$$

We may assume $|v(t) - u_0| \leq \epsilon$ on $[t_0, t_1]$ if $h > 0$ is chosen small enough. By induction, we obtain sequences (t_k), (u_k), (z_k) and (e_k) such that

$$\left.\begin{array}{c} t_k \nearrow t_\infty \leq a, \;\; u_k \in K_A(t_k), \;\; z_k = f(t_k, u_k), \\ e_k = u_{k+1} - S_{z_k}(t_{k+1} - t_k)u_k, \;\; |e_k| \leq \epsilon. \end{array}\right\} \tag{6}$$

For $k \geq 0$ we then let

$$v(t) = S_{z_k}(t - t_k)u_k + (t - t_k)e_k \quad \text{on } [t_k, t_{k+1}], \tag{7}$$

and may assume $t_{k+1} - t_k \leq \epsilon$ as well as $|v(t) - u_k| \leq \epsilon$ on $[t_k, t_{k+1}]$ by appropriate choice of the t_k. Let $P = \{t_k : k \geq 0\}$ and define $w \in L^1([0, t_\infty]; X)$ by means of $w(t) := z_k$ if $t \in [t_k, t_{k+1})$ and $w(t_\infty) = 0$. We will show that

$$|v(t) - u(t; t_k, u_k, w)| \leq \epsilon(t - t_k) \quad \text{on } [t_k, t_\infty) \quad \text{for all } k \geq 0. \tag{8}$$

Notice that (8) implies $|v(t) - u(t; w)| \leq \epsilon t$ on $[0, t_\infty)$, hence (5) holds a.e. on $[0, t_\infty]$ by definition of w. Evidently, (8) holds if

$$|v(t) - u(t; t_k, u_k, w)| \leq \epsilon(t - t_k) \quad \text{on } [t_j, t_{j+1}] \tag{9}$$

for all $j \geq k \geq 0$ and (9) is valid for $j = k$, by construction of v. Suppose that (9) holds for fixed $k \geq 0$ and $j = m - 1 \geq k$. Exploitation of

$$u(t; t_k, u_k, w) = u(t; t_m, u(t_m; t_k, u_k, w), z_m) \quad \text{on } [t_m, t_{m+1}]$$

and

$$v(t) = u(t; t_m, u_m, z_m) + (t - t_m)e_m \quad \text{on } [t_m, t_{m+1}]$$

yields

$$\begin{array}{rcl} |v(t) - u(t; t_k, u_k, w)| & \leq & |u_m - u(t_m; t_k, u_k, w)| + (t - t_m)|e_m| \\ & \leq & (t_m - t_k)\epsilon + (t - t_m)\epsilon \end{array}$$

for all $t \in [t_m, t_{m+1}]$, hence (9) holds for $j = m$. By induction (9) is valid for all $j \geq k \geq 0$.

Since (8) implies

$$v([0, t_\infty)) \subset C_k + (t_\infty - t_k)\overline{B}_\epsilon(0) \quad \text{for all } k \geq 0,$$

where $C_k := v([0,t_k]) \cup u([t_k,t_\infty];t_k,u_k,w)$ is relatively compact, it follows that $v([0,t_\infty))$ is relatively compact. Let (u_{k_j}) be a convergent subsequence of $(u_k) = (v(t_k))$ and define $v(t_\infty) := \lim_{j\to\infty} u_{k_j}$. Then $v(t_\infty) \in K_A(t_\infty)$ since $\mathrm{gr}\,(K_A)$ is closed from the left, and therefore $v : [0,t_\infty] \to X$ has those properties required in the definition of M^ϵ. Moreover, it is easy to check that (8) is also valid on $[t_k,t_\infty]$, and therefore $(v,w,P,t_\infty) \in M^\epsilon$.

2. $M^\epsilon \neq \emptyset$ by step 1, and we shall use Zorn's lemma to obtain an element of M^ϵ with $b = a$. For this purpose define a partial ordering on M^ϵ by $(v,w,P,b) \le (\overline{v},\overline{w},\overline{P},\overline{b})$ if

$$b \le \overline{b},\ v = \overline{v} \text{ on } [0,b],\ w = \overline{w} \text{ a.e. on } [0,b],\ P \subset \overline{P}.$$

To be able to apply Zorn's lemma we have to show that every ordered subset $M \subset M^\epsilon$ has an upper bound in M^ϵ. Let

$$b^* = \sup\{b \in (0,a] : (v,w,P,b) \in M \text{ for some } v,w,P\}.$$

In case the "sup" is actually a "max", i.e. if there is $(v,w,P,b^*) \in M$, we let

$$P^* = \{\tau \in [0,b^*) :\ \text{there is } (v,w,P,b^*) \in M \text{ with } \tau \in P\}.$$

Evidently, (v,w,P^*,b^*) is an upper bound and $(v,w,P^*,b^*) \in M^\epsilon$ is easy to check.
In the remaining case there is a sequence $(v_n,w_n,P_n,b_n) \subset M$ with $b_n \nearrow b^*$, hence $P_n \subset P_{n+1}$, $v_{n+1} = v_n$ on $[0,b_n]$ and $w_{n+1} = w_n$ a.e. on $[0,b_n]$ for all $n \ge 1$. We then let

$$P^* = \bigcup_{n\ge 1} P_n, \quad v^*(t) = v_n(t) \text{ on } [0,b_n], \quad w^*(t) = w_n(t) \text{ on } [0,b_n].$$

Suppose, for the moment, that $v^*([0,b^*))$ is relatively compact. We let $v^*(b^*) = \lim_{j\to\infty} v^*(b_{n_j})$ where $(v^*(b_{n_j}))$ is a convergent subsequence of $(v^*(b_n))$, and claim that $(v^*,w^*,P^*,b^*) \in M^\epsilon$ is an upper bound for M. Evidently, (v^*,w^*,P^*,b^*) is an upper bound for M, since $(v,w,P,b) \in M$ implies $b < b_n$, hence $(v,w,P,b) \le (v_n,w_n,P_n,b_n)$ for some $n \ge 1$. To check that $(v^*,w^*,P^*,b^*) \in M^\epsilon$ is also easy; notice that $\tau \in P^*$ implies $\tau \in P_n$ and $v^*(\tau) = v_n(\tau)$ for all $n \ge n_\tau$. So, it remains to prove relative compactness of $v^*([0,b^*))$. But the latter follows by the corresponding arguments from step 1, where this time we take any sequence $(t_k) \subset P^*$ with $t_k \nearrow b^*$ and $u_k := v^*(t_k)$; notice that (8) then holds with v^* instead of v.
Consequently, there is a maximal element $(v^*,w^*,P^*,b^*) \in M^\epsilon$ and we are done if $b^* = a$. Suppose $b^* < a$. We then let $t_0 = b^*$, $u_0 = v^*(b^*)$ and repeat the construction of step 1 to obtain the sequences from (6) and function v from (7). Let

$$\overline{v}(t) = v^*(t) \text{ on } [0,b^*],\ \overline{v}(t) = v(t) \text{ on } [b^*,t_\infty),\ \overline{b} = t_\infty,$$
$$\overline{w}(t) = w^*(t) \text{ on } [0,b^*],\ \overline{w}(t) = z_k \text{ on } [t_k,t_{k+1}],\ \overline{P} = P^* \cup \{t_k : k \ge 0\}.$$

Then $\overline{v}([t_0,t_\infty))$ is relatively compact again, and, as before, we let $\overline{v}(t_\infty) := \lim_{j\to\infty} \overline{v}(t_{k_j})$ for an appropriate subsequence (t_{k_j}).

To obtain $(\overline{v},\overline{w},\overline{P},\overline{b}) \in M^\epsilon$ we show that $\tau \in P^*$ and $t \in (t_0, t_\infty)$ implies $|\overline{v}(t) - u(t;\tau,\overline{v}(\tau),w)| \le \epsilon(t-\tau)$; the other cases as well as the remaining properties are rather obvious. Due to (8) and the properties of (v^*, w^*, P^*, b^*) we have

$$\begin{aligned} & |\overline{v}(t) - u(t;\tau,\overline{v}(\tau),w)| \\ \le\ & |v(t) - u(t;t_0,u_0,w)| + |u(t;t_0,u_0,w) - u(t;t_0,u(t_0;\tau,v^*(\tau),w),w)| \\ \le\ & \epsilon(t-t_0) + |v^*(t_0) - u(t_0;\tau,v^*(\tau),w)| \le \epsilon(t-t_0) + \epsilon(t_0-\tau) = \epsilon(t-\tau), \end{aligned}$$

hence $(\overline{v},\overline{w},\overline{P},\overline{b}) \in M^\epsilon$ with $\overline{b} > b^*$, a contradiction. Consequently, $b^* = a$ for every maximal element of M^ϵ. □

Observe that f need not be continuous in Lemma 4.1. This fact will be important in case of multivalued perturbations later on.

4.2 Locally Lipschitz perturbations

In case f is locally Lipschitz continuous, the "subtangential condition" (3) is also sufficient for existence of a solution in $K(\cdot)$, provided that f satisfies the growth condition

$$|f(t,x)| \le c(1+|x|) \text{ on } \mathrm{gr}\,(K_A) \text{ with some } c > 0. \tag{10}$$

Theorem 4.1 *Let A be m-accretive in a real Banach space X, $J = [0,a] \subset \mathbb{R}$ and $K : J \to 2^X$ be such that $K_A(0) \neq \emptyset$ and $gr\,(K_A)$ is closed from the left. Let $f : gr\,(K_A) \to X$ be locally Lipschitz continuous, satisfying (3) and (10). Then (1) has a unique mild solution for every $u_0 \in K_A(0)$.*

Proof. It suffices to establish existence of a mild solution, since uniqueness is an obvious consequence of the local Lipschitz continuity.

1. Let $u_0 \in K_A(0)$ be fixed. To simplify subsequent arguments, we first reduce to the case when f is bounded on $\mathrm{gr}\,(K_A)$. For this purpose, let $r(\cdot)$ be the solution of

$$r' = 1 + c(1 + r + |S(t)u_0|) \quad \text{on } J, \quad r(0) = 0,$$

and define

$$\hat{K}(t) := K(t) \cap \overline{B}_{r(t)}(S(t)u_0) \quad \text{and } \hat{K}_A(t) := \hat{K}(t) \cap \overline{D(A)} \quad \text{for } t \in J.$$

Evidently $u_0 \in \hat{K}_A(0)$, $\mathrm{gr}\,(\hat{K}_A)$ is closed from the left and f is bounded on $\mathrm{gr}\,(\hat{K}_A)$. In order to show that (3) also holds for $\hat{K}$ instead of K, let $t \in [0,a)$, $x \in \hat{K}_A(t)$ and $z := f(t,x)$. Due to $z \in T_K^A(t,x)$ there are sequences $h_n \to 0+$ and $e_n \to 0$ such that

$$S_z(h_n)x + h_n e_n \in K_A(t+h_n) \quad \text{for all } n \ge 1.$$

By means of the estimate

$$|S_z(h_n)x + h_n e_n - S(t+h_n)u_0| \le |S_z(h_n)x - S(h_n)x| + |x - S(t)u_0| + h_n|e_n| \le$$
$$h_n|f(t,x)| + r(t) + h_n|e_n| \le r(t) + h_n c(1 + r(t) + |S(t)u_0|) + h_n|e_n| \;\le\; r(t+h_n),$$

which holds if $n \ge 1$ is sufficiently large, this implies

$$S_z(h_n)x + h_n e_n \in \hat{K}_A(t+h_n) \quad \text{for all large } n \ge 1,$$

hence (3) also holds for $\hat{K}$. Consequently, all assumptions of Theorem 4.1 are also satisfied if K is replaced by $\hat{K}$.

2. We are done if (1) admits a local mild solution. Indeed, if this holds we obtain a noncontinuable mild solution by means of Zorn's lemma, and this solution is necessarily defined on all of J since f is bounded on $\mathrm{gr}\,(\hat{K}_A)$.
Fix $\delta > 0$ such that f is Lipschitz (say of constant L) on $([0,\delta] \times \bar{B}_\delta(u_0)) \cap \mathrm{gr}\,(\hat{K}_A)$, and let $b \in (0,\delta]$ be such that $r(t) + |S(t)u_0 - u_0| \le \delta$ on $[0,b]$. Let $\tilde{K}$ be the restriction of $\hat{K}$ to $\tilde{J} = [0,b]$. Then f is Lipschitz of constant L on $\mathrm{gr}\,(\tilde{K}_A)$, since $\tilde{K}(t) = \hat{K}(t) \subset \bar{B}_{r(t)}(S(t)u_0) \subset \bar{B}_\delta(u_0)$ on $\tilde{J}$.
Consider $\epsilon_n \searrow 0$. Application of Lemma 4.1 yields $w_n \in L^1(\tilde{J};X)$ such that

$$w_n(t) \in f([\tilde{J}_{t,\epsilon_n} \times \overline{B}_{\gamma\epsilon_n}(u_n(t))] \cap \mathrm{gr}\,(\tilde{K}_A)) \quad \text{a.e. on } \tilde{J}$$

with $\gamma = 1 + b$ and $\tilde{J}_{t,\epsilon_n} = [t - \epsilon_n, t] \cap \tilde{J}$, where u_n is the mild solution of

$$u_n' + Au_n \ni w_n(t) \;\text{ on } \tilde{J}, \;\; u_n(0) = u_0.$$

The former inclusion implies $w_n(t) = f(\tau_n(t), v_n(t))$ a.e. on $\tilde{J}$ with certain function τ_n, v_n such that $v_n(t) \in \tilde{K}_A(\tau_n(t))$, $t - \epsilon_n \le \tau_n(t) \le t$ and $|u_n(t) - v_n(t)| \le \gamma\epsilon_n$ on $\tilde{J}$. Therefore

$$\begin{aligned} |w_n(t) - w_m(t)| &\le L(|\tau_n(t) - \tau_m(t)| + |v_n(t) - v_m(t)|) \\ &\le L|u_n(t) - u_m(t)| + L(1+\gamma)(\epsilon_n + \epsilon_m) \;\text{ a.e on } \tilde{J}. \end{aligned}$$

Consequently $\varphi(t) := |u_n(t) - u_m(t)|$ satisfies

$$\varphi(t) \le L\int_0^t \Big(\varphi(s) + (1+\gamma)(\epsilon_n + \epsilon_m)\Big)ds \;\text{ on } \tilde{J}, \;\; \varphi(0) = 0.$$

Application of Gronwall's lemma shows that (u_n) is Cauchy in $C(\tilde{J};X)$, hence $|u_n - u|_0 \to 0$ for some $u \in C(\tilde{J};X)$ with $u(0) = u_0$. Evidently $\tau_n(t) \to t-$ and $v_n(t) \to u(t)$ as $n \to \infty$ for every $t \in (0,b]$, which implies $u(t) \in \tilde{K}_A(t)$ on $\tilde{J}$ since $\mathrm{gr}\,(\tilde{K}_A)$ is closed from the left. The same argument shows that $w_n(t) = f(\tau_n(t), v_n(t)) \to f(t, u(t))$ a.e. on $\tilde{J}$, hence $w_n \to f(\cdot, u(\cdot))$ in $L^1(\tilde{J};X)$. Therefore u is a mild solution of (1) on $\tilde{J}$. □

In the situation of Theorem 4.1 the mild solution $u(\cdot;u_0)$ depends continuously on $u_0 \in K_A$; a

proof in case f is Carathéodory and locally Lipschitz in x will be given in Theorem 4.4 below. Let us note in passing that the necessary condition $K_A(t) \neq \emptyset$ on J is of course implicitely contained in the assumptions of Theorem 4.1. Nevertheless, we did not include this condition explicitely, since the reduction to bounded f becomes easier this way.
Theorem 4.1 immediately yields the following characterization of positive invariance.

Corollary 4.1 *Let A be m-accretive in a real Banach space X, $J = [0, a] \subset \mathbb{R}$, $K : J \to 2^X$ be such that $gr(K_A)$ is closed from the left and $f : J \times \overline{D(A)} \to X$ be locally Lipschitz satisfying (10). Then $K(\cdot)$ is positively invariant for $u' + Au \ni f(t, u)$ iff (3) is valid.*

In several applications it happens that for an appropriate choice of the $K(t)$ these sets are positively invariant for the resolvents of A. Then it is helpful to know that the subtangential condition can be separated, by which we mean that

$$J_\lambda K(t) \subset K(t) \text{ for } \lambda > 0, t \in [0, a) \text{ and } f(t, x) \in T_K(t, x) \text{ for } t \in [0, a), x \in K_A(t) \tag{11}$$

implies (3). We don't have a simple direct proof of this fact, but it is not difficult to show that (11) implies the "weak range condition"

$$\underline{\lim}_{h \to 0+} h^{-1} \rho(x + hf(t, x), (I + hA)(K(t + h) \cap D(A))) = 0 \text{ for } t \in [0, a), x \in K_A(t), \tag{12}$$

and the latter in turn implies (3). This is the contents of the next result which allows for continuous f.

Lemma 4.2 *Let A be m-accretive in a real Banach space X, $J = [0, a] \subset \mathbb{R}$, $K : J \to 2^X$ with $gr(K_A)$ closed from the left, and $f : gr(K_A) \to X$ be continuous. Then (11) implies (12) and the latter implies (3).*

Proof. 1. To establish the first implication, let $t \in [0, a)$ and $x \in K_A(t)$. Then, given $\epsilon > 0$, there is $h \in (0, \epsilon]$ and $e \in X$ with $|e| \le \epsilon$ such that $x + h(f(t, x) + e) \in K(t + h)$, hence

$$J_h(x + h(f(t, x) + e)) \in K(t + h) \cap D(A).$$

Consequently,

$$\rho(x + hf(t, x), (I + hA)(K(t + h) \cap D(A))) \le h\epsilon$$

and therefore (12) holds.

2. To obtain the second implication, let $t_0 \in [0, a)$ and $x_0 \in K_A(t_0)$. Evidently (3) holds if, given $\eta \in (0, 1]$, there is $\delta \in (0, \eta]$ such that

$$\rho(S_{f(t_0,x_0)}(\delta)x_0, K_A(t_0 + \delta)) \le 4\delta\eta. \tag{13}$$

Fix $\eta \in (0, 1]$ and let $r \in (0, \eta]$ with $t_0 + r \le a$ be such that

$$\sup\{|f(t_0, x_0) - f(t, x)| : t \in [t_0, t_0 + r], x \in K_A(t) \cap \overline{B}_r(x_0)\} \le \eta.$$

Choose $(x,y) \in \mathrm{gr}\,(A)$ such that $|x_0 - x| \le r/4$ and let $\sigma = \frac{1}{2}r(2+|f(t_0,x_0)|+|y|)^{-1}$. We are going to construct local ϵ-DS-approximate solutions for

$$u' + Au \ni f(t,u) \quad \text{on } [t_0, t_0+\sigma], \quad u(t_0) = x_0, \tag{14}$$

and to compare them to corresponding ϵ-DS-approximate solutions for

$$v' + Av \ni f(t_0,x_0) \quad \text{on } [t_0, t_0+\sigma], \quad v(t_0) = x_0. \tag{15}$$

Let $\epsilon \in (0,\eta]$ with $\epsilon \le \sigma/2$. Exploitation of (12) yields $h_k \in (0,\epsilon]$ and $e_k \in X$ with $|e_k| \le \epsilon$ such that

$$x_{k+1} := J_{h_k}(x_k + h_k(f(t_k,x_k)+e_k)) \in K_A(t_{k+1}) \quad \text{for } k \ge 0 \tag{16}$$

where $t_{k+1} := t_k + h_k$. Given these h_k we also let

$$\overline{x}_{k+1} := J_{h_k}(\overline{x}_k + h_k f(t_0,x_0)) \quad \text{for } k \ge 0, \quad \overline{x}_0 := x_0. \tag{17}$$

Since all J_{h_k} are nonexpansive it follows by induction that

$$\left.\begin{aligned} |x_k - \overline{x}_k| &\le (t_k - t_0)(\epsilon + \max_{j=1,\dots,k-1} |f(t_j,x_j) - f(t_0,x_0)|), \\ |\overline{x}_k - x_0| &\le (t_k - t_0)|f(t_0,x_0)| + |J_{h_{k-1}} \cdots J_{h_0} x_0 - x_0|. \end{aligned}\right\} \tag{18}$$

Application of Proposition 1.1(d) yields

$$|J_{h_{k-1}} \cdots J_{h_0} x_0 - x_0| \le 2|x_0 - x| + (t_k - t_0)|y|,$$

hence

$$|x_k - x_0| \le (t_k - t_0)(2 + |f(t_0,x_0)| + |y|) + 2|x_0 - x|$$

as long as $t_k - t_0 \le r$ and $|x_k - x_0| \le r$. By the choice of σ it follows that $|x_k - x_0| \le r$ for all $k \ge 1$ such that $t_k \le t_0 + \sigma$.

To obtain an ϵ-DS-approximate solution for (14) by means of (16), we have to show that the h_k can be chosen such that $t_m \ge t_0 + \delta$ for some $m \ge 1$. This can be achieved by the usual trick: For $t \in [0,a)$ and $x \in K_A(t)$ let

$$\varphi_\epsilon(t,x) = \sup\{h \in (0,\epsilon] : \rho(x + hf(t,x), (I+hA)(K(t+h) \cap D(A))) \le \epsilon h\},$$

and choose $h_k \ge \frac{1}{2}\varphi_\epsilon(t_k,x_k)$, say, in each step. Now suppose $t_k \nearrow t_\infty \le \delta$. Since (17) means $\overline{x}_{k+1} = J^z_{h_k}\overline{x}_k$ where J^z_λ is the resolvent of A_z with $z := f(t_0,x_0)$, the estimate in Proposition 1.1(d) shows that $(\overline{x}_k)$ is a Cauchy sequence. Hence

$$|x_{k+l} - x_k| \le (t_{k+l} - t_j)(\epsilon + 1) + (t_k - t_j)(\epsilon + 1) + |\overline{x}_{k+l} - \overline{x}_k|$$

for all $l \ge 1$, $k > j \ge 0$ (which follows again by induction) implies that (x_k) is a Cauchy sequence too. Consequently, $x_k \to x_\infty \in K_A(t_\infty)$ as $k \to \infty$ and therefore

$$\varliminf_{(t,x)\to(t_\infty-,x_\infty)} \varphi_\epsilon(t,x) \le \lim_{k\to\infty} \varphi_\epsilon(t_k,x_k) \le 2\lim_{k\to\infty} h_k = 0.$$

This is a contradiction, since we will show

$$\varliminf_{(s,y)\to(t-,x)} \varphi_\epsilon(s,y) > 0 \quad \text{for all } t\in[0,a),\ x\in K_A(t). \tag{19}$$

For this purpose, choose $h \geq \frac{1}{2}\varphi_{\epsilon/3}(t,x) > 0$ and $e \in B_{\epsilon/2}(0)$ such that

$$x + h(f(t,x)+e) \in (I+hA)(K(t+h)\cap D(A)).$$

Given $t_n \nearrow t$ and $x_n \in K_A(t_n)$ with $x_n \to x$, let $h_n = h + t - t_n \geq h$. Then

$$J_h(x+h(f(t,x)+e)) \in K(t+h)\cap D(A) = K(t_n+h_n)\cap D(A).$$

Using the resolvent identity and letting $z := x + h(f(t,x)+e)$, we get

$$J_h z = J_{h_n}\Big(z + \frac{t-t_n}{h}(z - J_h z)\Big),$$

hence

$$z + \frac{t-t_n}{h}(z-J_h z) \in (I+h_n A)(K(t_n+h_n)\cap D(A)) =: R_n$$

and therefore

$$\begin{aligned} &\rho(x_n + h_n f(t_n,x_n), R_n)\\ \leq\ & |x-x_n| + h|f(t,x)-f(t_n,x_n)| + (t-t_n)(|f(t_n,x_n)| + |z-J_h z|/h) + \epsilon\frac{h}{2}\\ \leq\ & \epsilon h \leq \epsilon h_n \end{aligned}$$

for all large $n\geq 1$, i.e. $\varliminf_{n\to\infty} \varphi_\epsilon(t_n,x_n) \geq h > 0$ and consequently (19) holds.

Thus we get ϵ-DS-approximate solutions u^ϵ, v^ϵ for (14), (15) having the values x_k, $\overline{x}_k$ on $[t_k,t_{k+1})$ for $k=0,\dots,m$, respectively, and $t_m < t_0+\sigma \leq t_{m+1}$. Since A_z (with $z = f(t_0,x_0)$) is m-accretive it holds that $v^\epsilon(t) \to S_z(t-t_0)x_0$ uniformly on $[t_0,t_0+\sigma]$ as $\epsilon\to 0+$. Therefore

$$|\overline{x}_k - S_z(t_k-t_0)x_0| \leq \sigma\eta \quad \text{for } k=0,\dots,m,$$

if $\epsilon>0$ is chosen sufficiently small. Moreover, by (16) and (18),

$$\rho(\overline{x}_k, K_A(t_k)) \leq (t_k-t_0)\Big(\epsilon + \sup\{|f(t_0,x_0)-f(t,x)| : t\in[t_0,t_0+r], x\in K_A(t)\cap \overline{B}_r(x_0)\}\Big)$$

for $k=0,...,m$; recall that $|x_k - x_0| \leq r$ for those k. Consequently,

$$\rho(S_z(t_m-t_0)x_0, K_A(t_m)) \leq \sigma\eta + 2(t_m-t_0)\eta$$

by the choice of r and ϵ. Finally $t_0+\sigma \leq t_{m+1} \leq t_m + \epsilon$ and $\epsilon \leq \sigma/2$ imply $\sigma \leq 2(t_m-t_0)$, hence (13) holds with $\delta = t_m - t_0$. □

Theorem 4.1 combined with Lemma 4.2 obviously implies

Corollary 4.2 *Let A be m-accretive in a real Banach space X, $J=[0,a]\subset\mathbb{R}$ and $K: J\to 2^X$ be such that $K_A(0)\neq\emptyset$ and $gr(K_A)$ is closed from the left. Let $f: gr(K_A)\to X$ be locally Lipschitz continuous, satisfying (10). Then (1) has a unique mild solution for every $u_0\in K_A(0)$ if, in addition, (11) or (12) is fulfilled.*

4.3 Continuous perturbations

Given again a tube K, let $f : \mathrm{gr}\,(K_A) \to X$ be continuous such that the necessary condition (3) holds. We consider initial value problem (1) under additional compactness assumptions similar to those in §3 and, since existence of approximate solutions is guaranteed by Lemma 4.1, it is rather obvious that (1) admits a mild solution if $-A$ generates a compact semigroup. On the other hand, there are several applications in which (also due to the choice of K) the perturbation f has the additional property that

$$f : \mathrm{gr}\,(K_A) \to X \text{ maps bounded sets into weakly relatively compact sets.} \tag{20}$$

In this situation the compactness assumption on A can be weakened somewhat: instead of a compact semigroup it suffices that

$$\begin{array}{c}\mathcal{S} : L^1(J;X) \to C(J;X) \text{ maps weakly relatively} \\ \text{compact sets into relatively compact sets.}\end{array} \tag{21}$$

Recall that $\mathcal{S}w(= u(\cdot\,;w))$ denotes the mild solution of (4) with $t_0 = 0$ where $u_0 \in K_A(0)$ is fixed, and notice that (21) holds if $-A$ generates a compact semigroup due to Lemma 3.1. Although property (21) looks rather technical, it nevertheless can be verified in concrete cases where the semigroup lacks compactness; see §6.1 below.

Theorem 4.2 *Let A be m-accretive in a real Banach space X, $J = [0,a] \subset \mathbb{R}$ and $K : J \to 2^X$ be such that $K_A(0) \neq \emptyset$ and $gr(K_A)$ is closed from the left. Let $f : gr(K_A) \to X$ be continuous, satisfying (10) such that one of (3), (11) or (12) holds. Then (1) has a mild solution for every $u_0 \in K_A(0)$, if also one of the following assumptions is fulfilled.*
(a) $-A$ generates a compact semigroup.
(b) f satisfies (20) and A is such that (21) holds.

Proof. By the first step of the proof of Theorem 4.1 we may assume that f is bounded on $\mathrm{gr}\,(K_A)$. Given $\epsilon_n \searrow 0$, application of Lemma 4.1 eventually together with Lemma 4.2 yields $w_n \in L^1(J;X)$ such that

$$w_n(t) \in f([J_{t,\epsilon_n} \times \overline{B}_{\gamma\epsilon_n}(u_n(t))] \cap \mathrm{gr}\,(K_A)) \quad \text{a.e. on } J$$

with $\gamma = 1 + a$ and $J_{t,\epsilon_n} = [t - \epsilon_n, t] \cap J$, where $u_n = \mathcal{S}w_n$.
If (a) holds then (u_n) is relatively compact in $C(J;X)$ by means of Lemma 3.1 since (w_n) is even bounded in $L^\infty(J;X)$. To obtain the same conclusion in case of (b), observe that

$$w_n(t) \in f([J \times B] \cap \mathrm{gr}\,(K_A)) \quad \text{a.e. on } J,$$

where $B := \{u_n(t) : t \in J, n \geq 1\} + \overline{B}_{\gamma\epsilon_1}(0)$ is bounded. Therefore (w_n) is weakly relatively compact in $L^1(J;X)$ by Lemma 3.2 and (20), hence (u_n) is relatively compact in $C(J;X)$

by (21). We may therefore assume $u_n \to u$ in $C(J;X)$. Then $u(t) \in K_A(t)$ on J as well as $w_n \to f(\cdot, u(\cdot))$ in $L^1(J;X)$ follows as in the proof of Theorem 4.1, hence u is a mild solution of (1). $\square$

Throughout the rest of this section we consider continuous perturbations of compact type. Compared to §3, we shall impose the stronger condition

$$\lim_{h\to 0+} \beta(f([J_{t,h} \times B] \cap \mathrm{gr}\,(K_A)) \le k(t)\beta(B) \ \text{ a.e. on } J \text{ for all bounded } B \subset X \tag{22}$$

with $J_{t,h} = [t-h, t] \cap J$ and $k \in L^1(J)$. Notice that (22) is the appropriate assumption in order to handle with the deviation in t that appears in the inclusion (5). For the same reason it is not possible to obtain directly an integral inequality for $\varphi(t) = \beta(\{u_n(t) : n \ge 1\})$, where u_n are approximate solutions according to Lemma 4.1. To overcome the latter problem we use the following standard result from differential inequalities. For convenience we include the short proof.

Proposition 4.1 *Let $J = [0,a] \subset \mathbb{R}$ and $\varphi : J \to \mathbb{R}_+$ be measurable with $\varphi(0) = 0$, satisfying*

$$D^-\varphi(t) \le k(t)\varphi(t) \ \text{ a.e. on } J, \quad \varphi(t) \le \varphi(s) + M(t-s) \ \text{ for } 0 \le s \le t \le a, \tag{23}$$

with $k \in L^1(J)$ and $M > 0$, where $D^-\varphi(t)$ denotes the upper left Dini-derivative.
Then $\varphi(t) = 0$ for every $t \in J$.

Proof. Extend φ and k to $[0, a+1]$ by means of $\varphi(t) = \varphi(a)$ and $k(t) = 0$ for $t > a$. Then (23) still hold if J is replaced by $[0, a+1]$. Given $\epsilon \in (0,1]$, consider $\varphi_\epsilon(t) = \frac{1}{\epsilon}\int_0^\epsilon \varphi(t+\tau)d\tau$ on J. Then φ_ϵ is absolutely continuous with $\varphi_\epsilon(t) \to \varphi(t)$ a.e. on J as $\epsilon \to 0+$. In addition,

$$\varphi_\epsilon'(t) = \lim_{h\to 0+} \frac{\varphi_\epsilon(t) - \varphi_\epsilon(t-h)}{h} \le \frac{1}{\epsilon}\int_0^\epsilon \psi_\delta(t+\tau)d\tau \ \text{ a.e. on } J \text{ for } \delta > 0$$

with

$$\psi_\delta(s) := \max\{0, \sup_{0<h\le\delta} \frac{\varphi(s) - \varphi(s-h)}{h}\} \to \max\{0, D^-\varphi(s)\} \ \text{ as } \delta \to 0+\,.$$

Since $\psi_\delta(\cdot) \le M$ by (23), the dominated convergence theorem implies

$$\varphi_\epsilon'(t) \le \frac{1}{\epsilon}\int_0^\epsilon k(t+\tau)\varphi(t+\tau)d\tau \le \frac{M(a+1)}{\epsilon}\int_0^\epsilon |k(t+\tau) - k(t)|d\tau + k(t)\varphi_\epsilon(t) \ \text{ a.e. on } J.$$

Application of Gronwalls' lemma, together with $\varphi_\epsilon(0) \le M\epsilon$ by (23), shows that

$$\varphi_\epsilon(t) \le M\epsilon e^{|k|_1} + M(a+1)e^{|k|_1}\frac{1}{\epsilon}\int_0^\epsilon\int_0^t |k(s+\tau) - k(s)|ds\,d\tau \ \text{ on } J,$$

hence $\varphi_\epsilon(t) \to 0$ on J as $\epsilon \to 0+$. Consequently $\varphi(t) = 0$ a.e. on J, hence (23) yields $\varphi(t) = 0$ for every $t \in J$. $\square$

We are now able to obtain

Theorem 4.3 *Let A be m-accretive in a real Banach space X, $J = [0,a] \subset \mathbb{R}$ and $K : J \to 2^X$ be such that $K_A(0) \neq \emptyset$ and $gr(K_A)$ is closed from the left. Let $f : gr(K_A) \to X$ be continuous, satisfying (10) and (22) such that one of (3), (11) or (12) holds. Then (1) has a mild solution for every $u_0 \in K_A(0)$, if also one of the following assumptions is fulfilled.*
(a) X^* is uniformly convex and $-A$ generates an equicontinuous semigroup.
(b) $A = A_0 + g$ with $D(A) := D(A_0)$, where A_0 is linear, densely defined and m-accretive, $g : X \to X$ is continuous, accretive.

Proof. Let $u_0 \in K_A(0)$ be given. As before we may assume that f is bounded and, given $\epsilon_n \searrow 0$, we obtain approximate solutions $u_n = \mathcal{S}w_n$ where (w_n) is bounded in $L^\infty(J;X)$ such that

$$w_n(t) \in f([J_{t,\epsilon_n} \times \overline{B}_{\gamma\epsilon_n}(u_n(t))] \cap \mathrm{gr}\,(K_A)) \quad \text{a.e. on } J.$$

Moreover, we are done if (u_n) is relatively compact in $C(J;X)$.

Let X_0 be a closed separable subspace of X such that $w_n(t) \in X_0$ a.e. on J and $u_n(t) \in X_0$ on J for all $n \geq 1$. We let $\Omega = \overline{D(A)} \cap X_0$ and $\varphi(t) = \beta_\Omega(\{u_n(t) : n \geq 1\})$ on J. Then $\varphi : J \to \mathbb{R}_+$ is measurable with $\varphi(0) = 0$. We claim that

$$\varphi(t) \leq \varphi(s) + 2\int_s^t \beta_{X_0}(\{w_n(\tau) : n \geq 1\})d\tau \quad \text{for } 0 \leq s \leq t \leq a. \tag{24}$$

Fix $s,t \in J$ with $s < t$ and let $r = \varphi(s)$. Given $\epsilon > 0$, there are $x_1, \dots, x_m \in \Omega$ (for some $m \geq 1$) such that $\{u_n(s) : n \geq 1\} \subset \bigcup_{i=1}^m B_{r+\epsilon}(x_i)$. Let $N_i = \{n \geq 1 : u_n(s) \in B_{r+\epsilon}(x_i)\}$. For fixed $i \in \{1, \dots m\}$ and $n \in N_i$ let v_n denote the mild solution of

$$v_n' + Av_n \ni w_n(\tau) \text{ on } [s,t], \quad v_n(s) = x_i.$$

Evidently $|u_n(t) - v_n(t)| \leq |u_n(s) - x_i| \leq r + \epsilon$ for all $n \in N_i$, hence

$$\beta_\Omega(\{u_n(t) : n \in N_i\}) \leq r + \epsilon + \beta_\Omega(\{v_n(t) : n \in N_i\}) \leq r + \epsilon + 2\beta(\{v_n(t) : n \in N_i\}).$$

If (a) holds then application of Lemma 3.7 to (v_n) yields

$$\beta_\Omega(\{u_n(t) : n \in N_i\}) \leq r + \epsilon + 2\int_s^t \beta_{X_0}(\{w_n(\tau) : n \geq 1\})d\tau,$$

and in case of (b) the same inequality follows by means of (22) in the proof of Theorem 3.5. Consequently,

$$\varphi(t) = \max_{i=1,\dots,m} \beta_\Omega(\{u_n(t) : n \in N_i\}) \leq \varphi(s) + 2\int_s^t \beta_{X_0}(\{w_n(\tau) : n \geq 1\})d\tau + \epsilon.$$

Hence (24) holds since $\epsilon > 0$ was arbitrary. Exploitation of (24) and (22) yields

$$D^-\varphi(t) \leq 4k(t)\varphi(t) \quad \text{a.e. on } J, \tag{25}$$

which can be seen as follows. Evidently (24) implies $D^-\varphi(t) \le 2\psi(t)$ a.e. on J with $\psi(s) = \beta_{X_0}(\{w_n(s) : n \ge p\})$; notice that ψ is measurable and independent of $p \ge 1$. Fix $t \in J$ such that (22) holds, let $\eta > 0$ and $B = \{u_n(t) : n \ge 1\} + B_r(0)$ with $r > 0$. Since B is bounded there is $h > 0$ such that

$$\beta(f([J_{t,h} \times B] \cap \mathrm{gr}\,(K_A))) \le k(t)\beta(B) + \eta.$$

Hence $\{w_n(t) : n \ge p\} \subset f([J_{t,h} \times B] \cap \mathrm{gr}\,(K_A))$ for all large $p \ge 1$ implies

$$\psi(t) \le 2\beta(\{w_n(t) : n \ge p\}) \le 2[k(t)\beta(B) + \eta] \le 2[k(t)\varphi(t) + k(t)r + \eta].$$

The latter holds for all $\eta, r > 0$, hence $\psi(t) \le 2k(t)\varphi(t)$ and therefore (25) is valid.
Now recall that $|w_n(t)| \le M$ a.e. on J with some $M > 0$. Hence (24) yields

$$\varphi(t) \le \varphi(s) + 2M(t-s) \quad \text{for all } 0 \le s \le t \le a.$$

Consequently, the assumptions of Proposition 4.1 are fulfilled (with $2M, 4k$ instead of M, k) and therefore $\varphi(t) = 0$ on J. Hence $(u_n) \subset C(J; X)$ has relatively compact sections and then equicontinuity of (u_n) follows, as before, from

$$|u_n(t) - u_n(\bar{t})| \le |S(|t-\bar{t}|)u_n(s) - u_n(s)| + M(|t-s| + |\bar{t}-s|) \quad \text{for } 0 \le s \le t, \bar{t} \le a,$$

where $S(t)$ denotes the semigroup generated by $-A$. Thus (u_n) is relatively compact which ends the proof. □

4.4 Carathéodory perturbations

If f is only strongly measurable with respect to t, the approximate solutions guaranteed by Lemma 4.1 are not helpful due to the deviation in t. We are able to overcome this problem in the important special case when the tube $K(\cdot)$ is replaced by a fixed closed set K. Consequently, we consider (1) with Carathéodory $f : J \times K_A \to X$, where we assume that

$$|f(t,x)| \le c(t)(1 + |x|) \quad \text{on } J \times K_A \text{ with } c \in L^1(J). \tag{26}$$

In this situation the following modification of Lemma 4.1 leads to approximate solutions that are adapted to the Carathéodory case.

Lemma 4.3 *Let A be m-accretive in a real Banach space X, $J = [0,a] \subset \mathbb{R}$ and $K : J \to 2^X$ with closed values be increasing with respect to inclusion and such that $K_A(0) \neq \emptyset$. Let $f : gr\,(K_A) \to X$ be bounded such that $f(\cdot, x)$ is continuous from the right and (3) is satisfied. Then, given $u_0 \in K_A(0)$ and $\epsilon > 0$, there is $w \in L^1(J; X)$ such that*

$$w(t) \in f(t, \overline{B}_{\gamma\epsilon}(u(t;w)) \cap K_A(t)) \quad \text{a.e. on } J \tag{27}$$

with $\gamma = 1 + a$.

Proof. Since the proof parallels the one given for Lemma 4.1, it suffices to explain the difference in the construction of the approximate solutions. Let $u_0 \in K_A(0)$ and $\epsilon \in (0,1]$. We consider the set M^ϵ again, but with (27) instead of (5). The basic idea in order to obtain (27) on a first interval $[0,h]$, is to replace $S_{f(0,u_0)}(\cdot)u_0$ by the mild solution $v_0(\cdot)$ of

$$v_0' + Av_0 \ni f(t,u_0) \;\text{ on } J,\;\; v_0(0) = u_0.$$

This initial value problem obviously admits a mild solution if $g := f(\cdot,u_0) : J \to X$ is strongly measurable, and this holds if continuity from the right implies strong measurability. For the sake of completeness we include a short proof of this fact which belongs to "folklore". Fix $\eta > 0$. Then, due to continuity of g from the right, for every $t \in J$ there is $\delta(t) > 0$ such that $|g(t) - g(s)| \le \eta$ for $s \in [t, t+\delta(t)] \cap J$. Since

$$J = \bigcup_{t\in J} \bigcup_{0<h\le\delta(t)} [t,t+h]$$

is a Vitali cover of J, application of Vitali's covering theorem (see p.262ff in Hewitt/Stromberg [66]) yields $t_k \in J$ and $h_k \in (0,\delta(t_k)]$ such that the $[t_k, t_k+h_k]$ are pairwise disjoint and $J_0 := \bigcup_{k\ge1} [t_k, t_k+h_k]$ satisfies $\lambda_1(J \setminus J_0) = 0$. Then $g_\eta : J \to X$, defined by $g_\eta(t) = h(t_k)$ on $[t_k, t_k+h_k]$ and $g_\eta(t) = 0$ on $J \setminus J_0$, is a step function with $|g(t) - g_\eta(t)| \le \eta$ a.e. on J. Therefore g is strongly measurable.

Consequently, the initial value problem above has mild solution v_0, and

$$\frac{1}{h}|v_0(h) - S_{f(0,u_0)}(h)u_0| \le \frac{1}{h}\int_0^h |f(t,u_0) - f(0,u_0)|dt \to 0 \;\text{ as } h \to 0+,$$

since $f(\cdot,u_0)$ is continuous from the right. Hence there is $h_0 \in (0,\epsilon]$ such that

$$\frac{1}{h}|v_0(h) - S_{f(0,u_0)}(h)u_0| \le \frac{1}{3}\epsilon \;\text{ for all } h \in (0,h_0].$$

Since $z_0 := f(0,u_0) \in T_K^A(0,u_0)$, there is $h \in (0,h_0]$ such that $y_1 := S_{z_0}(h)u_0$ satisfies $\rho(y_1, K_A(h)) \le \frac{1}{2}\epsilon h$, hence $\rho(v_0(h), K_A(h)) < \epsilon h$. Choose $u_1 \in K_A(h)$ such that $|e_0| \le \epsilon$ for $e_0 := (u_1 - y_1)/h$. Let $t_0 = 0$, $t_1 = t_0 + h$ and

$$v(t) = v_0(t) + (t - t_0)e_0 \quad \text{on } [t_0,t_1],$$

where we may assume $|v(t) - u_0| \le \epsilon$ on $[t_0,t_1]$. By induction, the same construction yields $t_k \nearrow t_\infty \le a$, $u_k \in K_A(t_k)$ and mild solutions v_k of

$$v_k' + Av_k \ni f(t,u_k) \;\text{ on } [t_k,t_{k+1}],\;\; v_k(t_k) = u_k$$

such that $e_k := u_{k+1} - v_k(t_{k+1})$ satisfy $|e_k| \le \epsilon$. We then let

$$v(t) = v_k(t) + (t - t_k)e_k \;\text{ on } [t_k,t_{k+1}]$$

and may assume $t_{k+1} - t_k \le \epsilon$ as well as $|v(t) - u_k| \le \epsilon$ on $[t_k, t_{k+1}]$. Finally, we define $w \in L^1(J;X)$ by means of $w(t) = f(t, u_k)$ on $[t_k, t_{k+1}]$ and $w(t_\infty) = 0$.
Now, the arguments given in the proof of Lemma 4.1 show that $(v, w, \{t_k : k \ge 1\}, t_\infty) \in M^\epsilon$, where (27) follows from

$$w(t) = f(t, u_k) \in f(t, \overline{B}_\epsilon(v(t)) \cap K_A(t_k)) \subset f(t, \overline{B}_{\gamma\epsilon}(u(t;w)) \cap K_A(t)) \text{ on } [t_k, t_{k+1}].$$

Hence $M^\epsilon \ne \emptyset$, and a repetition of step 2 of the proof of Lemma 4.1 implies the existence of a maximal element of the type (v, w, P, a) of M^ϵ. □

By means of Lemma 4.3 and a reduction to the almost continuous case, we are able to obtain approximate solutions for Carathéodory $f : J \times K_A \to X$. From the preceeding sections it is then rather clear which additional properties are sufficient for convergence of an appropriate sequence of approximate solutions. The next result refers to situations in which we also get uniqueness. It includes the case when $f : J \times K_A \to X$ is locally Lipschitz with respect to x, by which we mean that for every $x_0 \in K_A$ there exist $\delta > 0$ and $\omega \in L^1(J)$ such that

$$|f(t,x) - f(t,\overline{x})| \le \omega(t)|x - \overline{x}| \text{ for a.a. } t \in J \text{ and all } x, \overline{x} \in B_\delta(x_0) \cap K_A.$$

Now we have

Theorem 4.4 *Let A be m-accretive in a real Banach space X, $J = [0,a] \subset \mathbb{R}$ and $K \subset X$ closed with $K_A = K \cap \overline{D(A)} \ne \emptyset$. Let $f : J \times K_A \to X$ be Carathéodory such that (26) and one of (3), (11) or (12) holds. Then (1) has a unique mild solution $u(\cdot\,; u_0)$ for every $u_0 \in K_A$, if also one of the following assumptions is fulfilled.*
(a) f is locally Lipschitz with respect to x.
(b) X^* is uniformly convex and $f(t,\cdot)$ is $\omega(t)$-dissipative with $\omega \in L^1(J)$.
In addition, this mild solution depends continuously on $u_0 \in K_A$.

Proof. Since it is easy to check that both (a) and (b) imply uniqueness of mild solutions, it suffices to prove existence and continuous dependence.

1. Let us first reduce to separable X. The arguments are similar to those in the proof of Theorem 3.2, but here we also have to care about the subtangential condition.
Given $u_0 \in K_A$, let $X_0 = \text{span}\{u_0\}$. By induction, we define an increasing sequence of closed separable subspaces as follows. Given a separable subspace X_n with $u_0 \in X_n$, let $M_n = \{y_k : k \ge 1\}$ be a dense subset of X_n such that $K_A \cap M_n$ is dense in $K_A \cap X_n$. Let $J_n \subset J$ with $\lambda_1(J \setminus J_n) = 0$ be such that $f(J_n \times (K_A \cap M_n))$ is contained in a separable subspace and $f(t,\cdot)$ is continuous for all $t \in J_n$. For every $k \ge 1$ choose $z_k \in K_A$ such that $|y_k - z_k| \le 2\rho(y_k, K_A)$, and let $K_n = \{z_k : k \ge 1\}$. Then X_{n+1} is defined as

$$X_{n+1} = \overline{\text{span}}\Big(X_n \cup K_n \cup f(J_n \times (K_A \cap X_n)) \cup \bigcup_{\lambda > 0} (I + \lambda A)^{-1} X_n\Big).$$

We then let $\hat{J} = \bigcap_{n\geq 0} J_n$, $\hat{X} = \overline{\bigcup_{n\geq 0} X_n}$ and $\hat{K} = \overline{\bigcup_{n\geq 0} (K_A \cap X_n)}$; notice that $\hat{K} \cap \overline{D(A)} = \hat{K}$. The arguments given in step 1 of the proof of Theorem 3.2 show that $\hat{J}$ is measurable with $\lambda_1(J \setminus \hat{J}) = 0$, $\hat{X}$ is a separable subspace, $f(\hat{J} \times \hat{K}) \subset \hat{X}$ and the restriction of A to $\hat{X}$ is m-accretive in $\hat{X}$. To complete the reduction, fix $\tau \in \hat{J}$ and redefine f on $(J \setminus \hat{J}) \times K_A$ by means of $f(t,x) := f(\tau, x)$.

Now notice that $f(t,x) \in T_K^A(x)$ for every $t \in [0,a)$ and $x \in K_A$, regardless of the subtangential condition that f originally satisfies. Indeed, if f satisfies (11) or (12) and $t_0 \in [0,a)$ is given, then application of Lemma 4.2 to g, defined by $g(t,x) := f(t_0, x)$ on $J \times K_A$, shows that $f(t_0,x) \in T_K^A(x)$ for every $x \in K_A$.

It remains to show that the restriction of f to $J \times \hat{K}$ satisfies the subtangential condition (3) with respect to $\hat{K}$. For this purpose, let us first show that

$$\rho(x, \hat{K}) \leq 2\rho(x, K_A) \ \text{ for every } x \in \hat{X}. \tag{28}$$

Fix $x \in \hat{X}$ and let $\epsilon > 0$. Then there is $y \in M_n$ for some $n \geq 0$ such that $|x - y| \leq \epsilon$. By definition of K_n there is $z \in K_n \subset K_A$ with $|y - z| \leq 2\rho(y, K_A)$. Hence $z \in K_A \cap X_{n+1} \subset \hat{K}$ and therefore

$$\rho(x, \hat{K}) \leq |x - z| \leq |x - y| + 2\rho(y, K_A) \leq 2\rho(x, K_A) + 3|x - y| \leq 2\rho(x, K_A) + 3\epsilon.$$

Hence (28) holds since $\epsilon > 0$ was arbitrary.
Let $t_0 \in J$ with $t_0 < a$, $x_0 \in \hat{K}$ and $v = f(t_0, x_0)$. Due to $v \in T_K^A(x_0)$ there is a sequence $h_m \to 0+$ such that

$$\frac{1}{h_m}\rho(S_v(h_m)x_0, K_A) \to 0 \ \text{ as } m \to \infty.$$

Since $x_0 \in \hat{K} \subset \hat{X} \cap \overline{D(A)}$, $v \in \hat{X}$ and A (restricted to $\hat{X}$) is m-accretive in $\hat{X}$ it follows that $S_v(h_m)x_0 \in \hat{X}$ for all $m \geq 1$. Hence (28) implies

$$\frac{1}{h_m}\rho(S_v(h_m)x_0, \hat{K}) \leq \frac{2}{h_m}\rho(S_v(h_m)x_0, K_A) \to 0 \ \text{ as } m \to \infty,$$

i.e. $f(t_0, x_0) \in T_{\hat{K}}^A(x_0)$.

2. By the previous step we may assume that X is separable, hence f is almost continuous by Lemma 3.4. Given $\epsilon > 0$, let $J_\epsilon \subset J$ be closed with $\lambda_1(J \setminus J_\epsilon) \leq \epsilon$ such that $f_{|J_\epsilon \times K_A}$ and $c_{|J_\epsilon}$ are continuous. We may assume $\{0, a\} \subset J_\epsilon$, have $J \setminus J_\epsilon = \bigcup_{k\geq 1} (a_k, b_k)$ with disjoint (a_k, b_k), and define

$$f_\epsilon(t,x) = \begin{cases} f(t,x) & \text{if } t \in J_\epsilon \\ f(a_k, x) & \text{if } t \in (a_k, b_k). \end{cases}$$

Then $f_\epsilon : J \times K_A \to X$ satisfies $|f_\epsilon(t,x)| \leq c_\epsilon(1 + |x|)$ on $J \times K_A$ with $c_\epsilon = \max_{J_\epsilon} c(t)$ and $f_\epsilon(\cdot, x)$ is continuous from the right. In order to apply Lemma 4.3, we consider f_ϵ on a smaller increasing tube $\hat{K}_A(\cdot)$ such that f_ϵ is bounded on $\mathrm{gr}\,(\hat{K}_A)$. For this purpose let $x_0 \in D(A)$ with $|u_0 - x_0| \le 1$, and recall that $S(\cdot)x_0$ is Lipschitz continuous of some constant L on J since x_0 belongs to the generalized domain of A. Let $r(\cdot)$ be the solution of

$$r' = L + c_\epsilon(1 + r + |S(t)x_0|) \text{ on } J, \;\; r(0) = 1$$

and $\hat{K}(t) = K(t) \cap \overline{B}_{r(t)}(S(t)x_0)$ on J. Evidently $u_0 \in \hat{K}_A(0)$, and the same arguments as given in step 1 of the proof of Theorem 4.1 show that f_ϵ satisfies $f_\epsilon(t,x) \in T^A_{\hat{K}}(t,x)$ for $(t,x) \in \mathrm{gr}\,(\hat{K}_A)$ with $t < a$. To see that $\hat{K}(\cdot)$ is increasing, let $0 \le s < t \le a$ and $x \in \hat{K}(s)$. Then

$$|x - S(t)x_0| \le |x - S(s)x_0| + L(t-s) \le r(s) + L(t-s) \le r(t)$$

shows that $x \in \hat{K}(t)$.

Consequently, application of Lemma 4.3 yields an approximate solution for (1) with f_ϵ instead of f. Given a sequence $\epsilon_n \searrow 0$, the considerations above provide corresponding sets J_n, functions f_n such that $f_n(t,x) = f(t,x)$ on $J_n \times K_A$ and approximate solutions $u_n = \mathcal{S}w_n$ with $w_n \in L^1(J;X)$ satisfying

$$w_n(t) \in f_n(t, \overline{B}_{\gamma\epsilon_n}(u_n(t)) \cap K_A) \;\text{ a.e on } J. \tag{29}$$

In addition, we may assume $J_n \subset J_{n+1}$ as well as

$$|f_n(t,x)| \le \hat{c}(t)(1 + |x|) \;\text{ on } J \times K_A \text{ for all } n \ge 1 \text{ with } \hat{c} \in L^1(J);$$

remember the construction given at the end of the proof of Theorem 3.2. By means of this growth condition there is $R > 0$ such that $|u_n|_0 \le R$, hence $|w_n(t)| \le \psi(t) := \hat{c}(t)(1+R)$ a.e. on J for all $n \ge 1$.

3. If (b) holds, we get relative compactness of (u_n) as follows. Due to (29) the w_n satisfy $w_n(t) = f(t, v_n(t))$ a.e. on J with $v_n(t) \in K_A$ such that $|u_n - v_n|_\infty \le \gamma\epsilon_n$. In particular, this implies

$$|u_n(t) - u_m(t)|^2 \le 2\int_{[0,t]\cap J_p} (f(s,v_n(s)) - f(s,v_m(s)), u_n(s) - u_m(s))_+ ds + 2\int_{J\setminus J_p} \psi(s)ds \;\text{ on } J$$

whenever $m, n \ge p \ge 1$. Let $\eta > 0$ be given. Since X^* is uniformly convex, $|u_n|_0 \le R$ and $|u_n - v_n|_\infty \le \gamma\epsilon_n$ there is $n_0 \ge 1$ such that

$$||\mathcal{F}(u_n(s) - u_m(s)) - \mathcal{F}(v_n(s) - v_m(s))|| \le \eta \quad \text{for all } m, n \ge n_0.$$

By means of a simple computation this yields

$$(f(s,v_n(s)) - f(s,v_m(s)), u_n(s) - u_m(s))_+$$

$$\le 2\omega(s)(|u_n(s) - u_m(s)|^2 + 4\gamma^2\epsilon_p^2) + 2\hat{c}(s)(1 + R + \gamma\epsilon_p)\eta \;\text{ on } J$$

for all $m, n \geq p$ if $p \geq n_0$. Consequently $\varphi(t) = |u_n(t) - u_m(t)|^2$ satisfies

$$\varphi(t) \leq 4\int_0^t \omega(s)\varphi(s)ds + 16\gamma^2|\omega|_1\epsilon_p^2 + 4(1 + R + \gamma\epsilon_p)|\hat{c}|_1\eta + 2\int_{J\setminus J_p} \psi(s)ds \ \text{ on } J$$

for $m, n \geq p$ with large p, hence application of Gronwall's lemma shows that (u_n) is Cauchy in $C(J; X)$. Consequently $u_n \to u$ in $C(J; X)$ and $u(t) \in K_A$ on J. Moreover, $w_n \to f(\cdot, u(\cdot))$ in $L^1(J; X)$ and therefore u is a mild solution of (1).

4. Let us now assume that (a) holds. Then there is $\delta > 0$ and $\omega \in L^1(J)$ such that $f(t, \cdot)$ is Lipschitz of constant $\omega(t)$ on $B_\delta(u_0) \cap K_A$. Due to $|w_n(t)| \leq \psi(t)$ a.e on J we find $b > 0$ such that $|u_n(t) - u_0| \leq \delta/2$ on $[0, b]$ for all $n \geq 1$. Exploitation of (29) yields $v_n : J \to K_A$ with $|u_n - v_n|_\infty \leq \gamma\epsilon_n$ such that $w_n(t) = f(t, v_n(t))$ a.e. on J, hence in particular $v_n(t) \in B_\delta(u_0) \cap K_A$ on $[0, b]$ for all $n \geq n_0$. Fix $p \geq n_0$ and consider $m, n \geq p$. Then the Lipschitz continuity of $f(s, \cdot)$ implies

$$\begin{aligned}
|u_n(t) - u_m(t)| &\leq \int_0^t |w_n(s) - w_m(s)|ds \\
&\leq \int_{[0,t]\cap J_p} |f(s, v_n(s)) - f(s, v_m(s))|ds + 2\int_{J\setminus J_p} \psi(s)ds \\
&\leq \int_0^t \omega(s)|u_n(s) - u_m(s)|ds + 2\gamma|\omega|_1\epsilon_p + 2\int_{J\setminus J_p} \psi(s)ds \ \text{ on } [0, b].
\end{aligned}$$

Application of Gronwall's lemma shows that $(u_{n|[0,b]})$ is Cauchy in $C([0, b]; X)$, and the limit is obviously a mild solution of (1) on $[0, b]$. Consequently, (1) has a unique noncontinuable mild solution u. In case u is defined on $[0, \tau)$ only, exploitation of (26) shows that $\lim_{t\to\tau-} u(t)$ exists and then u has an extension to $[0, \tau + b]$ with $b > 0$ by the arguments from above with J replaced by $[\tau, a]$. This contradiction shows that u is a mild solution of (1) on J.

5. It remains to show that $u(\cdot\,; u_0)$ depends continuously on $u_0 \in K_A$, and if (b) holds this follows immediately from the dissipativity property of f. If (a) is satisfied, let $(x_n) \subset K_A$ with $x_n \to x_0$ and $u_n = u(\cdot\,; x_n)$ for $n \geq 0$. By simple modifications of the arguments given in step 4 it is clear that $u_n(t) \to u_0(t)$ uniformly on $[0, b]$ for some $b > 0$, and then either $u_n \to u_0$ in $C(J; X)$ or there is $\tau \in (0, a]$ such that $u_n(t) \to u_0(t)$ uniformly on $[0, \tau']$ for all $\tau' \in (0, \tau)$ but not on $[0, \tau]$. If the second case occurs we obtain $f(t, u_n(t)) \to f(t, u_0(t))$ a.e. on $[0, \tau)$, hence $f(\cdot, u_n(\cdot)) \to f(\cdot, u_0(\cdot))$ in $L^1([0, \tau]; X)$ by the dominated convergence theorem since $|u_n|_0 \leq R$ for all $n \geq 0$ with some $R > 0$. This implies $u_{n|[0,\tau]} \to u_{0|[0,\tau]}$ in $C([0, \tau]; X)$, a contradiction. □

Existence of a mild solution of (1) with Carathéodory $f : J \times K_A \to X$ can of course also be obtained under the usual compactness assumptions. Actually, instead of compactness of the semigroup $S(t)$, it suffices to assume that $S(t)_{|K_A}$ is compact, i.e. $S(t)B$ is relatively compact for all $t > 0$ and bounded $B \subset K_A$. Depending on the set K, this may eventually be a much weaker assumption. Concerning perturbations of compact type, notice that the improved inclusion (27) allows to replace (22) by

$$\beta(f(t,B)) \le k(t)\beta(B) \;\text{ a.e. on } J \text{ for all bounded } B \subset K_A \tag{30}$$

with $k \in L^1(J)$.

Theorem 4.5 *Let A be m-accretive in a real Banach space X, $J = [0,a] \subset \mathbb{R}$ and $K \subset X$ closed with $K_A = K \cap \overline{D(A)} \neq \emptyset$. Let $f : J \times K_A \to X$ be Carathéodory such that (26) and one of (3), (11) or (12) holds. Then (1) has a mild solution for every $u_0 \in K_A$, if also one of the following assumptions is fulfilled.*

(a) $-A$ generates a semigroup $S(t)$ such that $S(t)_{|K_A}$ is compact.

(b) X^* is uniformly convex, $-A$ generates an equicontinuous semigroup, f satisfies (30).

(c) $A = A_0 + g$ with $D(A) := D(A_0)$, where A_0 is linear, densely defined and m-accretive, $g : X \to X$ is continuous, accretive and f satisfies (30).

Proof. We follow the proof of Theorem 4.4 up to the end of step 2. We may then assume that X is separable, and obtain approximate solutions $u_n = \mathcal{S}w_n$ such that $|u_n|_0 \le R$, $|w_n(t)| \le \psi(t)$ a.e. on J with $\psi \in L^1(J)$ and (29) holds. It is also clear that we are done if (u_n) is relatively compact in $C(J;X)$.

Assume that (a) holds. Due to (29) there are $v_n : J \to K_A$ such that $|u_n(t) - v_n(t)| \le \gamma\epsilon_n$ on J; in particular $\{v_n(t) : n \ge 1\} \subset K_A$ is bounded for all $t \in J$. Consequently,

$$|u_n(t) - S(h)v_n(t-h)| \le \gamma\epsilon_n + \int_{t-h}^{t} \psi(s)ds \quad \text{for } 0 \le t-h \le t \le a,$$

implies

$$\begin{aligned}\beta(\{u_n(t) : n \ge 1\}) &= \beta(\{u_n(t) : n \ge p\}) \\ &\le \gamma\epsilon_p + \int_{t-h}^{t} \psi(s)ds \quad \text{for all } p \ge 1 \text{ and } 0 \le t-h < t \le a.\end{aligned}$$

Therefore (u_n) has relatively compact sections, and then equicontinuity of (u_n) follows as usual.

In the situations described in (b) or (c), application of Lemma 3.7, respectively of the estimate (22) in the proof of Theorem 3.5, yields

$$\beta(\{u_n(t) : n \ge 1\}) \le \int_0^t \beta(\{w_n(s) : n \ge 1\})ds \;\text{ on } J;$$

recall that X is separable. Hence (29) and (30) imply

$$\beta(\{u_n(t) : n \ge 1\}) \le \int_0^t k(s)(\beta(\{u_n(s) : n \ge 1\})ds + |k|_1\gamma\epsilon_p \;\text{ on } J \text{ for all } p \ge 1,$$

which shows that (u_n) has relatively compact sections. This ends the proof, since the latter implies equicontinuity of (u_n), again. □

4.5 Upper semicontinuous perturbations

We consider the initial value problem

$$u' \in -Au + F(t,u) \ \text{ on } J, \ \ u(0) = u_0 \tag{31}$$

with m-accretive A and a multivalued perturbation F of usc type. As before we assume that F is defined on $\mathrm{gr}\,(K_A)$ where $K : J \to 2^X$ is a given tube such that $K_A(t) := K(t) \cap \overline{D(A)} \neq \emptyset$ on J and $\mathrm{gr}\,(K_A)$ is closed from the left. In this situation the natural version of the subtangential condition (3) is given by

$$F(t,x) \cap T_K^A(t,x) \neq \emptyset \quad \text{for all } (t,x) \in \mathrm{gr}\,(K_A) \text{ with } t < a. \tag{32}$$

Let us note that (32) is necessary for existence of a solution in the special case $A = 0$ if F is usc with compact values. We also impose the growth condition

$$||F(t,x)|| \le c(1 + |x|) \ \text{ on } \mathrm{gr}\,(K_A). \tag{33}$$

The purpose of the present section is to explain how much can be obtained by means of the methods and tools from §3 and the preceeding sections, where we concentrate on the case when $-A$ generates a compact semigroup.

Theorem 4.6 *Let X be a real Banach space with uniformly convex dual, A an m-accretive operator in X such that $-A$ generates a compact semigroup, $J = [0,a] \subset \mathbb{R}$ and $K : J \to 2^X$ be such that $K_A(0) \neq \emptyset$ and $gr\,(K_A)$ is closed from the left. Let $F : gr\,(K_A) \to 2^X \setminus \emptyset$ be ϵ-δ-usc with closed convex values such that (32) and (33) are valid. Then (31) has a mild solution for every $u_0 \in K_A(0)$.*

Proof. Let $u_0 \in K_A(0)$ be given. By means of the arguments given in step 1 of the proof of Theorem 4.1, there is a tube $\hat{K}(\cdot) \subset K(\cdot)$ with $u_0 \in \hat{K}_A(0)$ such that F is bounded on $\mathrm{gr}\,(\hat{K}_A)$ and (32) holds with $\hat{K}$ instead of K. We may therefore assume that F is bounded. For every $(t,x) \in \mathrm{gr}\,(K_A)$ with $t < a$ choose any element $f(t,x)$ in $F(t,x) \cap T_K^A(t,x)$ and let $f(a,\cdot) = 0$. Given $\epsilon > 0$, application of Lemma 4.1 yields an approximate solution $u = \mathcal{S}w$ such that $w \in L^1(J;X)$ satisfies (5). Given $\epsilon_n \searrow 0$ this leads to a sequence $(w_n) \subset L^1(J;X)$ such that

$$w_n(t) \in F([J_{t,\epsilon_n} \times \overline{B}_{\gamma\epsilon_n}(u_n(t))] \cap \mathrm{gr}\,(K_A)) \ \text{ a.e. on } J \tag{34}$$

with $\gamma = 1 + a$, where $J_{t,\epsilon_n} = [t - \epsilon_n, t] \cap J$ and $u_n = \mathcal{S}w_n$. Evidently (w_n) is even bounded in $L^\infty(J;X)$, hence we may assume $u_n \to u$ in $C(J;X)$ by Lemma 3.1. Then (34) implies

$$w_n(t) \in F([J_{t,\eta} \times \overline{B}_\eta(u(t))] \cap \mathrm{gr}\,(K_A)) \ \text{ a.e. on } J \text{ for all } n \ge n_\eta. \tag{35}$$

Since F is ϵ-δ-usc with weakly compact convex values it follows that $\{w_n(t) : n \geq 1\}$ is weakly relatively compact for almost all $t \in J$. Hence (w_n) is weakly relatively compact in $L^1(J;X)$ due to Lemma 3.2. Without loss of generality $w_n \rightharpoonup w$ in $L^1(J;X)$, and then (35) yields

$$w(t) \in \overline{\mathrm{conv}} F([J_{t,\eta} \times \overline{B}_\eta(u(t))] \cap \mathrm{gr}\,(K_A)) \text{ a.e. on } J$$

for all $\eta > 0$. Exploiting again the fact that F is ϵ-δ-usc this implies $w(t) \in F(t, u(t))$ a.e. on J, since the $F(t,x)$ are also convex.
Now recall from the proof of Theorem 3.1(b) that $w_n \rightharpoonup w$ in $L^1(J;X)$ and $\mathcal{S}w_n \to u$ in $C(J;X)$ implies $u = \mathcal{S}w$ if X^* is uniformly convex. Consequently u from above is a mild solution of (31). □

By means of Example 3.1 we know that (31) need not have a mild solution, even in a finite dimensional Banach space in case of a compact semigroup and without constraints. On the other hand, it is of course worth to note conditions that ensure existence of a mild solution in general Banach spaces. Let us just mention one possible setting: The approximate solutions u_n obtained in the proof of Theorem 4.6 are of course bounded, hence

$$w_n(t) \in F([J \times B] \cap \mathrm{gr}\,(K_A)) \text{ a.e. on } J \text{ with bounded } B \subset X.$$

Consequently, the arguments given above also yield a mild solution of (31) if F is as in Theorem 4.6 and maps bounded subsets of $\mathrm{gr}\,(K_A)$ into weakly relatively compact sets, and

$$\left.\begin{array}{c} (w_n) \subset L^1(J;X) \text{ with } w_n(t) \in F(\mathrm{gr}\,(K_A)) \text{ a.e. on } J \\ \text{and } w_n \rightharpoonup w \text{ in } L^1(J;X) \text{ implies } \mathcal{S}w_n \to \mathcal{S}w. \end{array}\right\} \quad (36)$$

Let us record this modification of Theorem 4.6 for later use.

Theorem 4.7 *Let A be m-accretive in a real Banach space X, $J = [0,a] \subset \mathbb{R}$ and $K : J \to 2^X$ be such that $K_A(0) \neq \emptyset$ and $gr(K_A)$ is closed from the left. Let $F : gr(K_A) \to 2^X \setminus \emptyset$ be ϵ-δ-usc with weakly compact convex values satisfying (32), (33) and such that F maps bounded sets into weakly relatively compact sets. Given $u_0 \in K_A(0)$, initial value problem (31) has a mild solution if (36) holds.*

Let us note in passing that (36) can be replaced by "$w_n \rightharpoonup w$ in $L^1(J;X)$ such that $(w_n(t)) \subset C$ a.e. on J with weakly relatively compact C implies $\mathcal{S}w_n \to \mathcal{S}w$" in Theorem 4.7, and the latter automatically holds in the semilinear case $A = A_0 + g$ as considered for example in Theorem 4.5(c), if $-A$ generates a compact semigroup. This follows easily by means of Lemma 3.1 and the variation of constants formula. Let us also mention that, in this setting, $-A$ generates a compact semigroup if the same holds for $-A_0$ and g is bounded on bounded sets.

4.6 Remarks

Remark 4.1 Sections §4.1 - §4.3 are based on Bothe [21], while the results in §4.5 are from Bothe [22]. Concerning the fully nonlinear setting we are only aware of Bressan/Staicu [32] and Vrabie [111], besides the two references mentioned above. In Vrabie [111] problem (1) has been considered for m-accretive A with compact semigroup and continuous bounded $f : J \times K \to X$, where K is assumed to be "semi locally closed" which includes the case of locally closed K. In this situation a mild solution is obtained under a much stronger "subtangential condition". For closed K this condition essentially becomes

$$\lim_{h\to 0+} \sup\{h^{-1}\rho(S_{f(t,x)}(h)x, K_A) : (t,x) \in J \times K_A\} = 0,$$

and if the latter holds one immediately gets approximate solutions with constant step size. In Bressan/Staicu [32] existence of solutions in closed sets is obtained for multivalued lsc perturbation under the following assumptions: A is m-accretive in a real Banach space X such that $-A$ generates a compact semigroup, $K \subset \overline{D(A)}$ is closed and $F : [0,a] \times K \to 2^X \setminus \emptyset$ is lsc and bounded with closed convex values such that $F(t,x) \subset T_K^A(x)$ on $[0,a) \times K$. The proof exploits the fact that a bounded lsc F admits a certain type of "directionally continuous" selections f, and existence of a mild solution of (1) for such a right-hand side f is obtained via approximate solutions. The construction of these approximate solutions is related to Lemma 4.1, but relies on compactness of the semigroup.

In the semilinear case, i.e. when $-A$ generates a C_0-semigroup of bounded linear operators on X, the problem of existence of mild solutions on (locally) closed sets has been considered by many authors. Before we give some more details, let us briefly explain the relation to the m-accretive case. First of all, notice that all results of §4 remain valid for m-ω-accretive A, which follows from Remark 3.1 and the observation that $y \in T_K^A(t,x)$ implies $y + \omega x \in T_K^{A_\omega}(t,x)$ for $A_\omega = A + \omega I$. Now suppose that $A : D(A) \subset X \to X$ is a closed, linear and densely defined operator such that $-A$ generates a C_0-semigroup $S(t)$. In this situation there is $\omega \in \mathbb{R}$ and $M \geq 1$ such that $|S(t)| \leq Me^{-\omega t}$ on $\mathbb{R}_+$, and X can be equipped with an equivalent norm $||\cdot||$ to achieve $M = 1$. Then A is m-ω-accretive in $(X, ||\cdot||)$, hence we are within the framework of §4. Recall also that, in the semilinear case, $u \in C(J;X)$ is a mild solution of

$$u' + Au = f(t,u) \ \text{ on } J, \ \ u(0) = u_0 \tag{37}$$

iff u satisfies the variation of constants formula. Therefore, it is easy to see that the necessary subtangential condition (3) is equivalent to

$$\varliminf_{h\to 0+} h^{-1}\rho(S(h)x + hf(t,x), K(t+h)) = 0 \quad \text{for } (t,x) \in \text{gr}\,(K) \text{ with } t < a. \tag{38}$$

In Chapter VIII of Martin [78] existence of a (local) mild solution of (37) is obtained for continuous $f : J \times K \to X$, where $K \subset X$ is locally closed, in each of the following cases:

$f = g + h$ with continuous g, h where g is Lipschitz in x and h is compact; the semigroup is compact; the $f(t,\cdot)$ are L-dissipative. In each case a separated version of the subtangential condition is imposed; in particular $f(t,x) \in T_K(x)$ on $[0,a) \times K$. The first two results are special cases of Theorem 4.5 (notice that $g + h$ satisfies (30)), respectively of Theorem 4.2, while the dissipative case is related to Theorem 4.1(b) but not contained therein since we needed X^* to be uniformly convex.

In Pavel [90] it has been shown that the necessary condition (38) is sufficient to get a local mild solution of (37) for continuous $f : J \times K \to X$ in case of a compact semigroup. In Theorem 5.1.2 of [91] the same author considered multivalued perturbations in the semilinear case, where the assumptions are: X reflexive, compact semigroup, $K : J \to 2^X \setminus \emptyset$ a tube with locally closed graph and $F : \mathrm{gr}\,(K) \to 2^X \setminus \emptyset$ locally bounded and weakly usc with closed convex values such that (38) holds with $f(t,x)$ replaced by *any* $y \in F(t,x)$. This is essentially contained in Theorem 4.6, since "F ϵ-δ-usc" was used to get weak relative compactness of (w_n) and can be replaced by "F weakly usc" if X is reflexive.

The result mentioned above concerning dissipative perturbations has been extended in Iwamiya [72] to continuous $f : \mathrm{gr}\,(K) \to X$ (with $\mathrm{gr}\,(K)$ closed from the left) such that (38) and

$$(f(t,x) - f(t,\overline{x}), x - \overline{x})_- \le \omega(t, |x - \overline{x}|)|x - \overline{x}| \quad \text{for all } t \in J, x, \overline{x} \in K(t)$$

holds, where ω is a "uniqueness function" of Carathéodory type.

Let us also mention that extensions in a different direction have been given in Martin/Lightbourne [80], Amann [3] and Prüss [97] if $-A$ generates an analytic semigroup: in this case it is possible to obtain existence if, among other assumptions, f is only continuous with respect to some fractional power of A. Furthermore, the results given in these papers allow for time-dependent operators $A(t)$.

Finally, concerning the special case $A = 0$, let us just note that existence results for ordinary differential equations on (locally) closed sets can be found e.g. in Deimling [39] and Martin [78], while corresponding results for differential inclusions are given in Aubin/Cellina [8], Aubin [7] and Deimling [42]. Differential inclusions under time-dependent constraints have been studied in Bothe [18], [19], Frankowska/Plaskacz/Rzezuchowksi [55] and other references given there.

Remark 4.2 For dissipative (not necessarily continuous) $f : D(f) \subset X \to X$ and $K(t) \equiv K$ one may eventually apply invariance results for accretive operators to $A - f$. For example Theorem 2 in Pierre [95] implies that problem (1) with accretive A and s-dissipative f, considered as $u' + (A - f)u \ni 0$, has a mild solution if for every $x \in K_A := K \cap \overline{D(A)}$ and $\epsilon > 0$ there is $h \in (0, \epsilon]$, $x_h \in D(A) \cap D(f)$ and $y_h \in Ax_h$ such that

$$|x - x_h + h(f(x_h) - y_h)| \le h\epsilon \quad \text{and} \quad \rho(x_h, K_A) \le h\epsilon.$$

In case $D(f) = K$ this is just the weak range condition for $A - f$, and it becomes (12) if, in addition, f is continuous bounded and $\overline{K \cap D(A)} = K_A$.

Remark 4.3 In the situation of Theorem 4.6 or 4.7 it is not clear whether the separate conditions

$$J_\lambda K(t) \subset K(t) \text{ for } \lambda > 0, t \in [0, a), \quad F(t, x) \cap T_K(t, x) \neq \emptyset \text{ for } t \in [0, a), x \in K_A(t)$$

imply (32), without further assumptions. In the special case $K(t) \equiv K$ with closed convex $K \subset X$ this implication is valid. Indeed, given $t_0 \in [0, a)$, $x_0 \in K_A$ and $y \in F(t_0, x_0) \cap T_K(x_0)$, there is a continuous selection $f : K \to X$ of $T_K(\cdot)$ such that $f(x_0) = y$ by Michael's selection theorem, hence application of Lemma 4.2 shows that $f(x) \in T_K^A(x)$ on K_A and therefore $y \in T_K^A(x_0)$. In the time-dependent case $K(t)$, the same argument works under the strong extra assumption that $T_K(\cdot, \cdot)$ is lsc with convex values.

§5 Further Qualitative Results

By inspection of §3 it is obvious that the set $\mathcal{U}(u_0)$ of all mild solutions of

$$u' \in -Au + F(t,u) \ \text{ on } J = [0,a], \ \ u(0) = u_0 \tag{1}$$

is a compact subset of $C(J;X)$ in the situations considered there. Actually, $\mathcal{U}(u_0)$ is a compact R_δ-set, i.e. the intersection of a decreasing sequence of compact absolute retracts, as we are going to show now.

5.1 Structure of solution sets

The basic idea in order to prove that $\mathcal{U}(u_0)$ is a compact R_δ-set is to consider approximate problems where F in (1) is replaced by a decreasing sequence of appropriate $F_n \supset F$ such that $\mathcal{U}(u_0)$ is the intersection of the solution sets $\mathcal{U}_n$ corresponding to F_n. If all F_n admit locally Lipschitz selections then the $\mathcal{U}_n$ turn out to be contractible, and this is sufficient for our purpose since $\mathcal{U}(u_0)$ is a compact R_δ iff $\mathcal{U}(u_0)$ is the intersection of a decreasing sequence of compact contractible subsets of $C(J;X)$; see Hyman [71] for this characterization of compact R_δ-sets. Since relative compactness of $\mathcal{U}_n$ can only be obtained under very strong assumptions on F, we shall also use the following slight extension of the characterization just mentioned (see Lemma 5 in Bothe [23]).

Lemma 5.1 *Let Ω be a complete metric space and $\emptyset \neq B \subset \Omega$. Then B is a compact R_δ-set iff $B = \bigcap_{n\geq 1} B_n$ for a decreasing sequence of closed contractible B_n such that $\beta_0(B_n) \to 0$, where $\beta_0(\cdot)$ denotes the Hausdorff-measure of noncompactness in Ω.*

We concentrate on the situation described in Theorem 3.4, since it will then be clear how the same method of proof applies in the other cases.

Theorem 5.1 *Let X be a real Banach space with uniformly convex dual and A be m-accretive in X such that $-A$ generates an equicontinuous semigroup. Let $D = \overline{\text{conv}}\, D(A)$, $J = [0,a] \subset \mathbb{R}$ and $F : J \times D \to 2^X \setminus \emptyset$ with closed convex values satisfy*

$$||F(t,x)|| \leq c(t)(1+|x|) \quad \text{on } J \times \overline{D(A)} \text{ with } c \in L^1(J) \tag{2}$$

and

$$\beta(F(t,B)) \leq k(t)\beta(B) \ \ \text{a.e. on } J \text{ for all bounded } B \subset D \text{ with } k \in L^1(J). \tag{3}$$

Suppose that $F(\cdot,x)$ has a strongly measurable selection for every $x \in D$ and $F(t,\cdot)$ is weakly usc for almost all $t \in J$. Then $\mathcal{U}(u_0)$ is a compact R_δ-set in $C(J;X)$ for every $u_0 \in \overline{D(A)}$. In particular, $\mathcal{U}(u_0)$ is connected.

Proof. 1. Let $u_0 \in \overline{D(A)}$ and $\mathcal{U} = \mathcal{U}(u_0)$. To be able to apply Lemma 5.1, we use the following standard approximation of F by certain $F_n \supset F$. Let $r_n = 3^{-n}$, $(W_\lambda)_{\lambda\in\Lambda}$ be a locally finite refinement of the open covering $D := \overline{\text{conv}} D(A) \subset \bigcup_{x\in D} B_{r_n}(x)$ and $(\varphi_\lambda)_{\lambda\in\Lambda}$ be a locally Lipschitz continuous partition of unity subordinate to $(W_\lambda)_{\lambda\in\Lambda}$. For every $\lambda \in \Lambda$ let $x_\lambda \in D$ be such that $W_\lambda \subset B_{r_n}(x_\lambda)$ and define F_n by

$$F_n(t,x) = \sum_{\lambda\in\Lambda} \varphi_\lambda(x) C_\lambda(t) \quad \text{on } J \times D \qquad \text{with } C_\lambda(t) = \overline{\text{conv}}\, F(t, B_{2r_n}(x_\lambda)).$$

Then it is not difficult to show

$$F(t,x) \subset F_{n+1}(t,x) \subset F_n(t,x) \subset \overline{\text{conv}}\, F(t, B_{3r_n}(x) \cap D) \quad \text{on } J \times D \text{ for all } n \geq 1; \tag{4}$$

see Lemma 2.2 and the proof of Theorem 7.2 in Deimling [42]. Let $\mathcal{U}_n$ be the solution set of (1) with F_n instead of F. By (4) it is evident that $(\mathcal{U}_n)$ is a decreasing sequence such that $\mathcal{U} \subset \bigcap_{n\geq 1} \mathcal{U}_n$. We claim that $u_n \in \mathcal{U}_n$ for all $n \geq 1$ implies $u_{n_k} \to u \in \mathcal{U}$ for some subsequence (u_{n_k}) of (u_n). This yields $\mathcal{U} = \bigcap_{n\geq 1} \mathcal{U}_n$, but it also implies $\beta_0(\mathcal{U}_n) \to 0$, where $\beta_0(\cdot)$ denotes the Hausdorff-measure in $C(J;X)$; notice that $\rho_n := \sup_{v\in\mathcal{U}_n} \rho(v,\mathcal{U}) \to 0$ if the claim holds, hence $\mathcal{U}_n \subset \mathcal{U} + \overline{B}_{\rho_n}(0)$ yields $\beta_0(\mathcal{U}_n) \leq \rho_n \to 0$ since $\mathcal{U}$ is compact.
Given $u_n = \mathcal{S} w_n \in \mathcal{U}_n$ for $n \geq 1$ with $w_n \in F_n(\cdot, u_n(\cdot))$, we first get boundedness of (u_n), since all F_n satisfy

$$||F_n(t,x)|| \leq c(t)(2 + |x|) \quad \text{on } J \times D \tag{5}$$

by (2) and (4). Hence (w_n) is uniformly integrable and therefore (u_n) is equicontinuous due to Lemma 3.6(a). Let $\varphi(t) = \beta(\{u_n(t) : n \geq 1\})$ on J. Application of Lemma 3.7 yields

$$\varphi(t) \leq \int_0^t \beta(\{w_n(s) : n \geq p\})\, ds \quad \text{on } J \text{ for all } p \geq 1,$$

and exploitation of (3) and (4) implies

$$\beta(\{w_n(s) : n \geq p\}) \leq \beta(F(s, \{u_n(s) : n \geq p\} + B_{3r_p}(0)) \leq k(s)(\varphi(s) + 3r_p) \text{ a.e. on } J, \tag{6}$$

hence Gronwall's Lemma and $p \to \infty$ yields $\varphi(t) \equiv 0$. Consequently, $|u_{n_k} - u|_0 \to 0$ for some subsequence (u_{n_k}) and some $u \in C(J;X)$. Since (6) implies $\beta(\{w_n(s) : n \geq 1\}) = 0$ a.e. on J we may also assume $w_{n_k} \rightharpoonup w$ by Lemma 3.2. Then $w \in \text{Sel}\,(u)$ follows as in step 2 of the proof of Lemma 3.3, hence $u_{n_k} \to u \in \mathcal{U}$.

2. We are done if the $\overline{\mathcal{U}}_n$ are contractible, since then Lemma 5.1 applies with $B = \mathcal{U}$ and $B_n = \overline{\mathcal{U}}_n$. Fix $n \geq 1$, let g_λ be a strongly measurable selection of $F(\cdot, x_\lambda)$ for every $\lambda \in \Lambda$, and define f by

$$f(t,x) = \sum_{\lambda\in\Lambda} \varphi_\lambda(x) g_\lambda(t) \quad \text{on } J \times D.$$

Then $f(t,x) \in F_n(t,x)$ on $J \times D$ is obvious. Since $(W_\lambda)_{\lambda\in\Lambda}$ is locally finite, the $f(\cdot,x)$ are strongly measurable and for every $x_0 \in D$ there exist $\gamma > 0$ and $\delta > 0$ such that

$$|f(t,x) - f(t,\overline{x})| \leq \gamma c(t)|x - \overline{x}| \quad \text{for all } t \in J,\ x, \overline{x} \in B_\delta(x_0) \cap D \tag{7}$$

with $c(\cdot)$ from (2). Therefore, given $(\tau, x) \in J \times \overline{D(A)}$, the initial value problem

$$v' \in -Av + f(t,v) \quad \text{on } [\tau, a], \quad v(\tau) = x.$$

has a unique mild solution $v = v(\cdot;\tau,x)$ by Theorem 4.4 (applied with $K = D$) and v depends continuously on x; notice that (5) also holds for $|f(t,x)|$ instead of $||F_n(t,x)||$. Therefore

$$h(s,u)(t) = \begin{cases} u(t) & \text{if } t \in [0, sa] \\ v(t; sa, u(sa)) & \text{if } t \in (sa, a] \end{cases}$$

defines a function $h : [0,1] \times \overline{\mathcal{U}}_n \to \overline{\mathcal{U}}_n$ such that $h(0,u) = v(\cdot\,;0,u_0)$ and $h(1,u) = u$ on $\overline{\mathcal{U}}_n$. It remains to prove that h is continuous in order to obtain contractibility of $\overline{\mathcal{U}}_n$. For this purpose let $(s_k, u_k) \in [0,1] \times \overline{\mathcal{U}}_n$ with $(s_k, u_k) \to (s,u)$ where we may assume that (s_k) is monotone. Let $R > 0$ be such that $|u|_0 \leq R$ for all $u \in \mathcal{U}_n$ and let $z_k = h(s_k, u_k)$ as well as $z = h(s,u)$. Consider the case $s_k \nearrow s$. Then

$$|z_k(t) - z(t)| \leq |u_k - u|_0 + \int_{s_k a}^{sa} 2(2+R)c(\tau)d\tau \quad \text{on } [0, sa],$$

in particular $z_k(sa) \to z(sa)$. Hence $|z_k - z|_0 \to 0$, since $v(\cdot\,; sa, x)$ depends continuously on x. If $s_k \searrow s$ then

$$|z_k(t) - z(t)| \leq |u_k - u|_0 + \int_{sa}^{s_k a} 2(2+R)c(\tau)d\tau =: \alpha_k \quad \text{on } [0, s_k a].$$

Let $\gamma, \delta > 0$ be such that (7) holds for $x_0 = u(sa)$. Due to $z_k(sa) \to z(sa)$ there is $\eta > 0$ and $k_0 \geq 1$ such that $|z(t) - x_0| \leq \delta$ and $|z_k(t) - x_0| \leq \delta$ on $[sa, sa+\eta]$ for all $k \geq k_0$. Hence

$$|z_k(t) - z(t)| \leq \alpha_k + \gamma \int_{s_k a}^{t} c(\tau)|z_k(\tau) - z(\tau)|d\tau \quad \text{on } [s_k a, sa+\eta]$$

and therefore

$$|z_k(t) - z(t)| \leq \alpha_k \exp\Big(\gamma \int_{s_k a}^{sa+\eta} c(\tau)d\tau\Big) \quad \text{on } [0, sa+\eta] \text{ for all } k \geq k_0.$$

Continuous dependence of $v(\cdot\,; sa+\eta, x)$ on x implies $z_k(t) \to z(t)$ uniformly on $[sa+\eta, a]$, hence $|z_k - z|_0 \to 0$.

Consequently $\mathcal{U}$ is an R_δ-set, and $\mathcal{U}$ is connected since all $\overline{\mathcal{U}}_n$ have this property. □

The situation is more complicated if constraints are present. First of all, even in case $K(t) \equiv K$ one cannot expect a connected solution set if K is only closed; see Example 7.3 in Deimling

[42], where a two-dimensional initial value problem with two solutions is given. For closed convex K the situation is better, but further difficulties occur since we now need outer approximations F_n having locally Lipschitz selections f_n which also satisfy the subtangential condition with respect to K. Under the separate subtangential condition (12) such approximations are possible by means of

Lemma 5.2 *Let X be a Banach space, $\emptyset \neq D \subset X$ closed convex and $G : D \to 2^X \setminus \emptyset$ be ϵ-δ-usc with closed convex values such that $G(x) \cap T_D(x) \neq \emptyset$ on D. Then, given $\epsilon > 0$, there exists a locally Lipschitz $g_\epsilon : D \to X$ satisfying $g_\epsilon(x) \in T_D(x)$ as well as $g_\epsilon(x) \in G(B_\epsilon(x) \cap D) + B_\epsilon(0)$ on D.*

Proof. Let $\epsilon > 0$ be given and recall that $T_D(x) = \overline{\{\lambda(y-x) : \lambda > 0, y \in D\}}$ by Proposition 2.4 since D is closed convex. Hence $G(x) \cap T_D(x) \neq \emptyset$ on D implies that for every $x \in D$ there are $h_x > 0$ and $z_x \in X$ such that

$$z_x \in G(x) + B_{\epsilon/4}(0) \quad \text{and} \quad x + h_x z_x \in D.$$

Since G is ϵ-δ-usc, we also find $\gamma_x \in (0, \epsilon/4)$ such that

$$G(y) \subset G(x) + B_{\epsilon/2}(0) \quad \text{for all } y \in B_{2\gamma_x}(x) \cap D.$$

Let $(W_\lambda)_{\lambda \in \Lambda}$ be a locally finite refinement of the open covering $D \subset \bigcup_{x \in D} B_{\delta_x}(x)$ with $\delta_x = \min\{\gamma_x, h_x \gamma_x\}$, and $(\varphi_\lambda)_{\lambda \in \Lambda}$ be a locally Lipschitz partition of unity subordinate to $(W_\lambda)_{\lambda \in \Lambda}$. For every $\lambda \in \Lambda$ choose $x_\lambda \in D$ such that $W_\lambda \subset B_{\delta_{x_\lambda}}(x_\lambda)$ and denote h_{x_λ}, z_{x_λ} and γ_{x_λ} by h_λ, z_λ and γ_λ, respectively. Finally, let $f_\lambda(x) = \dfrac{1}{h_\lambda}(x_\lambda + h_\lambda z_\lambda - x)$ for $x \in D$ and define

$$g_\epsilon(x) = \sum_{\lambda \in \Lambda} \varphi_\lambda(x) f_\lambda(x) \quad \text{on } D.$$

Evidently g_ϵ is locally Lipschitz, and $g_\epsilon(x) \in T_D(x)$ on D since $f_\lambda(x) \in T_D(x)$ and $T_D(x)$ is convex. To obtain the other inclusion for g_ϵ, fix $x \in D$. Then

$$g_\epsilon(x) = \sum_{i=1}^{m} \varphi_{\lambda_i}(x) f_{\lambda_i}(x) \quad \text{with } \lambda_i \in \Lambda \text{ such that } \varphi_{\lambda_i}(x) > 0 \text{ for } i = 1, \ldots, m,$$

since the sum in the definition of g_ϵ is locally finite. Now $\varphi_{\lambda_i}(x) > 0$ implies $x_{\lambda_i} \in W_{\lambda_i}$, hence $|x - x_{\lambda_i}| < \delta_{\lambda_i} \leq \gamma_{\lambda_i}$. This yields

$$|f_{\lambda_i}(x) - z_{\lambda_i}| < \delta_{\lambda_i}/h_{\lambda_i} \leq \gamma_{\lambda_i} < \epsilon/4 \quad \text{as well as} \quad |x_{\lambda_i} - x_{\lambda_k}| < \gamma_{\lambda_i} + \gamma_{\lambda_k} \leq 2\gamma_{\lambda_k}$$

if we let $k \in \{1, \ldots, m\}$ be such that $\gamma_{\lambda_k} = \max\{\gamma_{\lambda_i} : i = 1, \ldots, m\}$. Consequently,

$$f_{\lambda_i}(x) \in B_{\epsilon/4}(z_{\lambda_i}) \subset G(x_{\lambda_i}) + B_{\epsilon/2}(0) \subset G(B_{2\gamma_{\lambda_k}}(x_{\lambda_k})) + B_{\epsilon/2}(0) \subset G(x_{\lambda_k}) + B_\epsilon(0)$$

and therefore $g_\epsilon(x) \in G(x_{\lambda_k}) + B_\epsilon(0) \subset G(B_\epsilon(x)) + B_\epsilon(0)$. $\square$

Lemma 5.2 is Theorem 2 in Bader/Kryszewski [10] and improves a similar approximation result (for single-valued continuous G) by the author (Lemma 1 in Bothe [25]), where it is in addition assumed that K is proximinal. Now a basic result is

Theorem 5.2 *Let X be a real Banach space with uniformly convex dual, A an m-accretive operator in X such that $-A$ generates a compact semigroup and $\emptyset \neq K \subset X$ be closed convex such that $J_\lambda K \subset K$ for $\lambda > 0$. Let $J = [0,a] \subset \mathbb{R}$ and $F : J \times K \to 2^X \setminus \emptyset$ be ϵ-δ-usc with closed convex values such that $||F(t,x)|| \leq c(1+|x|)$ and $F(t,x) \cap T_K(x) \neq \emptyset$ on $[0,a) \times K$. Then $\mathcal{U}(u_0) \neq \emptyset$ is a compact R_δ-set in $C(J;X)$ for every $u_0 \in K_A$.*

Proof. Let $u_0 \in K_A$, $\mathcal{U} = \mathcal{U}(u_0)$ and notice that $\mathcal{U} \neq \emptyset$ by Theorem 4.6 and Remark 4.3. Consider $\epsilon_n \searrow 0$. By Lemma 5.2, applied to $G(t,x) = \{0\} \times F(t,x)$ on $D = J \times K \subset \mathbb{R} \times X$, there are locally Lipschitz $f_n : J \times K \to X$ such that $f_n(t,x) \in T_K(x)$ on $[0,a) \times K$ and

$$f_n(t,x) \in F_n(t,x) := F(B_{\epsilon_n}(t,x) \cap [J \times K]) + B_{\epsilon_n}(0) \quad \text{on } J \times K.$$

Let $\mathcal{U}_n$ denote the solution set of (1) with F_n instead of F. Evidently $F_{n+1}(t,x) \subset F_n(t,x)$ on $J \times K$, hence the $\mathcal{U}_n$ form a decreasing sequence such that $\mathcal{U} \subset \bigcap_{n \geq 1} \mathcal{U}_n$. The growth condition on F implies that all $\mathcal{U}_n$ are bounded in $C(J;K_A)$, hence also relatively compact due to compactness of the semigroup. Therefore $\beta_0(\mathcal{U}_n) = 0$ as well as $\mathcal{U} = \bigcap_{n \geq 1} \mathcal{U}_n$, and contractibility of $\overline{\mathcal{U}_n}$ follows as before. $\square$

Let us note that the extra condition "X^* uniformly convex" can be dropped in case of single-valued continuous F. Additional information is given in Remark 5.1 below.

5.2 Periodic solutions and equilibria

Based on the previous results concerning (weak) positive invariance, we now provide sufficient conditions for existence of a T-periodic solution of the evolution problem

$$u' + Au \ni f(t,u) \quad \text{on } \mathbb{R}_+ \tag{8}$$

with m-accretive A and a T-periodic Carathéodory function $f : \mathbb{R}_+ \times K \to X$. Recall that the problem under consideration is equivalent to the periodic boundary value problem

$$u' + Au \ni f(t,u) \quad \text{on } J = [0,T], \quad u(0) = u(T), \tag{9}$$

which in turn is equivalent to the existence of a fixed point of the Poincaré operator P_T, where $P_T(u_0)$ is the set of all values $u(T)$ of the mild solutions of the initial value problem

$$u' + Au \ni f(t,u) \quad \text{on } J, \quad u(0) = u_0.$$

In the sequel we will always assume that K is closed bounded convex such that

$$|f(t,x)| \le c(t) \quad \text{on } J \times K \text{ with } c \in L^1(J) \tag{10}$$

and $S(t)K_A$ is relatively compact for $t > 0$, where $S(t)$ is the semigroup generated by $-A$ and $K_A = K \cap \overline{D(A)}$. Then $P_T(u_0)$ is nonempty for every $u_0 \in K_A$ by Theorem 4.5 if f satisfies the subtangential condition

$$f(t,x) \in T_K^A(x) \quad \text{on } [0,T) \times K_A. \tag{11}$$

Moreover, it is easy to see that $P_T : K_A \to 2^{K_A} \setminus \emptyset$ is a compact map with closed graph, but further information concerning K_A and especially about the sets $P_T(u_0)$ is needed to apply known fixed point results. In case of jointly continuous f, the latter difficulty will be circumvented in the common way of approximating f by locally Lipschitz continuous f_ϵ, where the main point is to achieve that the subtangential condition is satisfied by f_ϵ, too.

In a general Banach space additional difficulties may occur, since K_A need not have the fixed point property for compact maps. Indeed (8) need not have a T-periodic solution as shown by the following counter-example.

Example 5.1 Let $X = \mathbb{R}^3$ with $||x|| = \max\{\sqrt{x_1^2 + x_2^2}, |x_3|\}$,

$$D(A) = \{x \in X : \sqrt{x_1^2 + x_2^2} = x_3\} \quad \text{and} \quad Ax \equiv \{(0,0)\} \times \mathbb{R} \text{ on } D(A).$$

Evidently $R(I + \lambda A) = X$ for all $\lambda > 0$ and the resolvents, given by $J_\lambda x = (x_1, x_2, \sqrt{x_1^2 + x_2^2})$, are nonexpansive. Hence A is m-accretive and $-A$ generates the compact semigroup $S(t) \equiv I_{|D(A)}$.
Let $K = \{x \in X : x_3 \in [a,b], x_1^2 + x_2^2 \le R^2\}$ with $0 < a < b < R$ and define $f : K \to X$ by $f(x) = (x_2, -x_1, 0)$. Of course f is Lipschitz continuous and bounded with $f(x) \in T_K(x)$ on K. Moreover, f satisfies $f(x) \in T_{K_A}(x)$ on K_A. Together with

$$K_A \subset \{x \in X : a^2 \le x_1^2 + x_2^2 \le b^2\} = R(I + \lambda A_{|K \cap D(A)}),$$

this implies

$$\varliminf_{h \to 0+} h^{-1} \rho(x + hf(x), R(I + hA_{|K\cap D(A)})) = 0 \quad \text{on } K_A,$$

i.e. f satisfies the weak range condition, in particular $f(x) \in T_K^A(x)$ on K_A by Lemma 4.2. Nevertheless, given any $T \in (0, 2\pi)$ the corresponding periodic problem has no T-periodic solution. Notice that the initial value problems

$$u' + Au \ni f(u) \quad \text{on } \mathbb{R}_+, \quad u(0) = u_0 \in K_A$$

have unique solutions staying in K_A, which implies $u_3(t)^2 = u_1(t)^2 + u_2(t)^2 \equiv u_{0,1}^2 + u_{0,2}^2$. Therefore every such solution satisfies $u' = (u_2, -u_1, 0)$, hence is periodic with minimal period $2\pi \ne T$. ◇

This problem cannot occur if the subtangential condition (11) is replaced by the separate assumptions

$$J_\lambda K \subset K \text{ for all } \lambda > 0, \quad f(t,x) \in T_K(x) \text{ on } [0,T) \times K, \tag{12}$$

since then K_A turns out to be a retract of K. Moreover, in this situation the approximation result Lemma 5.2 applies in case of continuous f. We therefore have

Theorem 5.3 *Let A be m-accretive in a real Banach space X, $K \subset X$ closed bounded convex such that $J_\lambda K \subset K$ for $\lambda > 0$ and $S(t)K_A$ is relatively compact for $t > 0$, where $S(t)$ is the semigroup generated by $-A$ and $K_A = K \cap \overline{D(A)}$. Let $f : \mathbb{R}_+ \times K \to X$ be Carathéodory and T-periodic satisfying (10) and $f(t,x) \in T_K(x)$ for almost all $t \geq 0$ and all $x \in K$.*
Then the evolution problem (8) has a T-periodic mild solution.

Proof. 1. We start with the proof of Theorem 5.3 in case f is jointly continuous and bounded. Fix $\epsilon \in (0,1]$, let $J = [0,T]$, $D = J \times K$ and define $g : D \to \mathbb{R} \times X$ by $g(t,x) = (0, f(t,x))$. Application of Lemma 5.2 to $g : D \to \mathbb{R} \times X$, where $\mathbb{R} \times X$ is endowed with the maximum norm, yields a locally Lipschitz $g_\epsilon : D \to \mathbb{R} \times X$ such that $g_\epsilon = (\tau_\epsilon, f_\epsilon)$ where $f_\epsilon : J \times K \to X$ satisfies

$$f_\epsilon(t,x) \in T_K(x) \quad \text{and} \quad f_\epsilon(t,x) \in f(B_\epsilon(t,x) \cap [J \times K]) + B_\epsilon(0) \quad \text{on } J \times K. \tag{13}$$

We claim that the approximate periodic problem

$$u' + Au \ni f_\epsilon(t,u) \text{ on } J, \;\; u(0) = u(T) \tag{14}$$

admits a mild solution. Due to Theorem 4.1, the initial value problem

$$u' + Au \ni f_\epsilon(t,u) \text{ on } J, \;\; u(0) = u_0$$

has a unique mild solution $u_\epsilon = u_\epsilon(\cdot\,; u_0)$ for every $u_0 \in K_A$, and this solution depends continuously on $u_0 \in K_A$. Define $P_T^\epsilon : K_A \to K_A$ by $P_T^\epsilon u_0 = u_\epsilon(T; u_0)$. To show that P_T^ϵ has a fixed point, observe first that $P_T^\epsilon(K_A)$ is relatively compact. Indeed, since f is bounded there is $\gamma > 0$ such that $|f_\epsilon|_\infty \leq \gamma$ (uniformly for $\epsilon \in (0,1]$), hence

$$P_T^\epsilon(K_A) \subset M_\gamma := \bigcap_{h \in (0,T]} \{u(h; u_0, w) : u_0 \in K_A, w \in L^\infty([0,h]; X) \text{ with } |w|_\infty \leq \gamma\}$$

due to invariance of K_A. Evidently $M_\gamma \subset S(h)K_A + \overline{B}_{h\gamma}(0)$ for all $h \in (0,T]$, which implies relative compactness of M_γ, since $S(h)K_A$ is relatively compact for $h > 0$.

Let $\lambda(x) = \rho(x, \overline{D(A)})$ on K and define $R : K \to X$ by

$$Rx = \begin{cases} J_{\lambda(x)}x & \text{on } K \setminus \overline{D(A)} \\ x & \text{on } K \cap \overline{D(A)}. \end{cases}$$

Evidently $R_{|K_A} = I_{|K_A}$, and $J_\lambda K \subset K \cap D(A)$ for $\lambda > 0$ yields $R(K) \subset K_A$. By means of the resolvent identity it is easy to see that R is continuous, hence $R : K \to K_A$ is a retraction. Therefore, application of Schauder's fixed point theorem to $P_T^\epsilon \circ R : K \to K_A$ yields a fixed point of P_T^ϵ, hence a solution of (14).

Consider $\epsilon_n \searrow 0$. Due to the arguments given above there are mild solutions $u_n(\cdot)$ of the approximate problem (14) with $\epsilon = \epsilon_n$. To obtain relative compactness of (u_n) in $C(J;X)$, notice that $\{u_n(t) : n \geq 1\} \subset M_\gamma$ for all $t \in (0,T]$ if T is replaced by t in the definition of M_γ above. Hence $\{u_n(t) : n \geq 1\}$ is relatively compact for every $t \in J$, since $u_n(0) = u_n(T)$. This implies equicontinuity of (u_n) in the usual way, since

$$|u_n(t) - u_n(\bar{t})| \leq |S(|t-\bar{t}|)u_n(s) - u_n(s)| + \gamma(|t-s| + |\bar{t}-s|) \ \text{ for } 0 \leq s \leq t, \bar{t} \leq T.$$

We may therefore assume $u_n \to u$ in $C(J;X)$. Given $\eta > 0$, exploitation of (13) together with continuity of f yields

$$f_{\epsilon_n}(t, u_n(t)) \subset f(B_{\epsilon_n}(t, u_n(t)) \cap [J \times K]) + B_{\epsilon_n}(0) \subset f(t, u(t)) + B_\eta(0)$$

for all $n \geq n_\eta$, hence $f_{\epsilon_n}(t, u_n(t)) \to f(t, u(t))$ on J. Consequently $f_{\epsilon_n}(\cdot, u_n(\cdot)) \to f(\cdot, u(\cdot))$ in $L^1(J;X)$ and therefore $u(\cdot)$ is a mild solution of $u' + Au \ni f(t,u)$ on J such that $u(T) = \lim\limits_{n\to\infty} u_n(T) = \lim\limits_{n\to\infty} u_n(0) = u(0)$, i.e. $u(\cdot)$ is a mild solution of (9).

2. In the general case we repeat the reduction to separable X given in step 1 of the proof of Theorem 4.4, and may then assume that f is almost continuous by Lemma 3.4. Consequently, given $\epsilon > 0$, there exists a closed $J_\epsilon \subset J$ with $\lambda_1(J \setminus J_\epsilon) \leq \epsilon$ such that $f_{|J_\epsilon \times K}$ and $c_{|J_\epsilon}$ are continuous, $f(t,x) \in T_K(x)$ on $J_\epsilon \times K$ and $\{0, T\} \subset J_\epsilon$. As before, we exploit the fact that $J \setminus J_\epsilon = \bigcup\limits_{k \geq 1} (\alpha_k, \beta_k)$ with disjoint (α_k, β_k), and define $f_\epsilon : J \times K \to X$ by

$$f_\epsilon(t,x) = \begin{cases} f(t,x) & \text{if } t \in J_\epsilon \\ \dfrac{\beta_k - t}{\beta_k - \alpha_k} f(\alpha_k, x) + \dfrac{t - \alpha_k}{\beta_k - \alpha_k} f(\beta_k, x) & \text{if } t \in (\alpha_k, \beta_k). \end{cases} \tag{15}$$

Obviously f_ϵ is bounded and jointly continuous, and f_ϵ satisfies $f_\epsilon(t,u) \in T_K(u)$ on $J \times K$ since the $T_K(u)$ are convex. Therefore the periodic problem (14) with f_ϵ given by (15) has a mild solution by the previous step.

To finish the proof consider $\epsilon_n \searrow 0$ with $\sum\limits_{n\geq 1} \epsilon_n < \infty$. For every $n \geq 1$ let f_n be given by (15) for $\epsilon = \epsilon_n$, let J_n denote the corresponding set J_{ϵ_n} and u_n be a mild solution of (14) with f_n instead of f_ϵ. Now recall that the J_n can be chosen in such a way that, in addition,

$$|f_n(t,x)| \leq \hat{c}(t) \ \text{ on } J \times K_A \text{ for all } n \geq 1 \text{ with } \hat{c} \in L^1(J); \tag{16}$$

remember the construction given in the proof of Theorem 3.2. We may then repeat the arguments given in step 1 with M_γ replaced by

$$\bigcap_{h\in(0,T]} \{u(h;u_0,w) : u_0 \in K_A, w \in L^1([0,h];X) \text{ with } |w(t)| \leq \hat{c}(T-h+t) \text{ a.e. on } [0,h]\},$$

to obtain a convergent subsequence of (u_n), the limit of which is a solution of (9). □

Applied to the autonomous case $f : K \to X$, Theorem 5.3 implies the existence of a stationary solution, i.e. we have

Corollary 5.1 *Let A be m-accretive in a real Banach space X, $K \subset X$ closed bounded convex such that $J_\lambda K \subset K$ for $\lambda > 0$ and $S(t)K_A$ is relatively compact for $t > 0$, where $S(t)$ is the semigroup generated by $-A$ and $K_A = K \cap \overline{D(A)}$. Let $f : K \to X$ be continuous and bounded such that $f(x) \in T_K(x)$ on K. Then there exists $x \in D(A) \cap K$ such that $f(x) \in Ax$.*

Proof. By Theorem 5.3 the evolution problem

$$u' + Au \ni f(u) \quad \text{on } \mathbb{R}_+$$

has a periodic solution u_n of period $1/2^n$ for every $n \geq 1$, and the arguments given in the proof also show that $(u_{n|[0,1]})$ is relatively compact in $C([0,1];X)$. Given a convergent subsequence its limit $u(\cdot) \equiv x$ is a mild solution of $u' + Au \ni f(u)$. Evidently u is absolutely continuous and a.e. differentiable, hence u is also a strong solution by Theorem 1.2. This implies $x \in D(A)$ and $f(x) \in Ax$, i.e. the existence of a stationary solution. □

Let us provide some additional information concerning the case when X and X^* are uniformly convex. Of course Theorem 5.3 is applicable if the separated condition (12) holds, but here it is tempting to work with the necessary subtangential condition (11) since the difficulty concerning K_A disappears; notice that K_A is convex due to the fact that $\overline{D(A)}$ has this property since X is uniformly convex. Unfortunately, it is unclear how to approximate f by a locally Lipschitz f_ϵ, still keeping the subtangential condition (11). Here, another possibility is to work with the explicit subtangential condition

$$f(t,x) - Ax \subset T_K(x) \quad \text{for all } t \in [0,T), x \in K \cap D(A). \tag{17}$$

In order to obtain existence of (T-periodic) solutions of (8) within this situation, the crucial point is to show that (17) is sufficient for viability of K. This holds if K has nonempty interior, and is interesting in itself: observe that (17) is in fact a condition on $\partial K \cap D(A)$, only. Notice also that, depending on the contribution of A, condition (17) may lead to better results than (12). Actually, in the situation of Lemma 5.3 below, (17) is a necessary condition if A is single-valued; see Remark 5.3.

Lemma 5.3 *Let X be a real Banach space such that X and X^* are uniformly convex. Let A be m-accretive in X, $K \subset X$ closed convex with $\overset{\circ}{K} \cap D(A) \neq \emptyset$ and $f : [0,T] \times K \to X$ continuous. Then the explicit subtangential condition (17) implies (11).*

Proof. 1. Possibly after a shift we may assume $0 \in \overset{\circ}{K}$, hence $\overline{B}_\delta(0) \subset K$ for some $\delta > 0$. Fix $t_0 \in [0,T)$, let $\tilde{f}$ be a continuous extension of $f(t_0,\cdot)$ to all of X, $\tilde{f}_\epsilon$ be locally Lipschitz with $|\tilde{f} - \tilde{f}_\epsilon| \leq \frac{1}{2}\epsilon\delta$ and define g_ϵ by $g_\epsilon(x) = \tilde{f}_\epsilon(x) - 2\epsilon x$ on X. Then g_ϵ satisfies

$$g_\epsilon(x) + \mu\epsilon x - Ax \subset T_K(x) \text{ for all } x \in K \cap D(A) \text{ and } \mu \in [0,1]. \tag{18}$$

To see this, let $x \in K \cap D(A)$ and $y \in Ax$. By (17) there are sequences $h_n \to 0+$ and $e_n \to 0$ such that

$$x_n := x + h_n(f(t_0,x) - y + e_n) \in K \text{ for all } n \geq 1.$$

Moreover

$$x + h_n(g_\epsilon(x) + \mu\epsilon x - y + e_n) = (1 - (2-\mu)\epsilon h_n)x + h_n(f(t_0,x) - y + \epsilon z + e_n)$$

with $|z| \leq \delta/2$, hence

$$x + h_n(g_\epsilon(x) + \mu\epsilon x - y + e_n) = \gamma_n\Big[(1-\epsilon h_n)x_n + \epsilon h_n v_n\Big] + \epsilon h_n(1-\gamma_n)v_n$$

with

$$\gamma_n = \frac{1-(2-\mu)\epsilon h_n}{1-\epsilon h_n} \nearrow 1 \quad \text{and} \quad v_n = z + h_n(2-\mu)(f(t_0,x) - y + e_n).$$

Obviously $|v_n| \leq \delta$ for all large n, hence $x_n \in K$ together with $\overline{B}_\delta(0) \subset K$ and convexity of K implies $(1-\epsilon h_n)x_n + \epsilon h_n v_n \in K$ for those n. Consequently,

$$\gamma_n\Big[(1-\epsilon h_n)x_n + \epsilon h_n v_n\Big] \in K \text{ for all large } n,$$

hence

$$h_n^{-1}\rho(x + h_n(g_\epsilon(x) + \mu\epsilon x - y), K) \leq |e_n| + \epsilon(1-\gamma_n)|v_n| \to 0$$

and therefore (18) holds.

2. We claim that for every $u_0 \in K_A$ there exists $b = b(u_0) > 0$ such that the mild solution u of

$$u' + Au \ni g_\epsilon(u) \text{ on } [0,b], \;\; u(0) = u_0$$

satisfies $u(t) \in K$ on $[0,b]$. Once this is shown we are done, since then g_ϵ satisfies the necessary subtangential condition $g_\epsilon(x) \in T_K^A(x)$ on K_A, hence consideration of $\epsilon_n \searrow 0$ yields

$$f(t_0,x) = \lim_{n\to\infty} g_{\epsilon_n}(x) \in T_K^A(x) \quad \text{for every } x \in K_A$$

by closedness of $T_K^A(x)$.
Let $u_0 \in K_A$, choose $r > 0$ such that $g_{\epsilon|\overline{B}_r(u_0)}$ is Lipschitz, say of constant L, and define h_ϵ

by $h_\epsilon(x) = g_\epsilon(Rx)$ on X, where R is the radial retraction onto $\overline{B}_r(u_0)$. Then h_ϵ is Lipschitz of constant $\omega = 2L$, hence $\omega I - h_\epsilon$ is continuous and accretive. Due to Theorem 5.5 below it follows that the operator A_ϵ, given by $A_\epsilon x = Ax - h_\epsilon(x)$ on $D(A_\epsilon) = D(A)$, is such that $A_\epsilon + \omega I$ is m-accretive. Therefore the initial value problem

$$u' + A_\epsilon u \ni 0 \ \text{ on } \mathbb{R}_+, \ \ u(0) = u_0$$

has a mild solution u. It is easy to check that, for some $b > 0$, every mild solution of

$$v' + A_\epsilon v \ni 0 \ \text{ on } \mathbb{R}_+, \ \ v(0) = v_0 \in K_A \cap \overline{B}_{r/2}(u_0) \tag{19}$$

satisfies $|v(t) - u_0| \leq r$ on $[0, b]$. We show that $v(t) \in K$ on $[0, b]$ if v_0 also belongs to $D(A)$. Given such v_0, it follows by Theorem 1.3 that the corresponding mild solution v of (19) has a derivative $v'_+(t)$ from the right at every $t \geq 0$, $v'_+(\cdot)$ is continuous from the right and

$$v(t) \in D(A_\epsilon), \ \ v'_+(t) + A_\epsilon^0 v(t) = 0 \ \text{ for all } t \geq 0;$$

here $A_\epsilon^0 u$ denotes the unique element of minimal norm of $A_\epsilon u$. Notice that the equation above implies

$$v'_+(t) = -\Big(Av(t) - g_\epsilon(v(t))\Big)^0 \ \text{ on } [0, b),$$

since $|v(t) - u_0| \leq r$ on $[0, b]$.
Now suppose there is $\tau \in [0, b)$ and $\sigma > 0$ such that $v(\tau) \in \partial K$ and $v(t) \notin K$ on $(\tau, \tau + \sigma)$. Then $v(\tau + h) = v(\tau) + hv'_+(\tau) + o(h)$ as $h \to 0+$ together with (18) for $\mu = 0$ implies

$$v(\tau) + h_n(v'_+(\tau) + e_n) \in \partial K \quad \text{for all } n \geq 1 \text{ with } h_n \to 0+, \ e_n \to 0. \tag{20}$$

On the other hand, the choice $\mu = 1$ in (18) yields

$$v(\tau) + h_n(v'_+(\tau) + \epsilon v(\tau) + \tilde{e}_n) \in K \quad \text{for all } n \geq 1 \text{ with } \tilde{e}_n \to 0; \tag{21}$$

notice that the same choice of (h_n) is possible due to

$$T_K(x) = \{y \in X : \lim_{h \to 0+} h^{-1} \rho(x + hy, K) = 0\} \ \text{ for closed convex } K$$

by Proposition 2.4. To arrive at a contradiction, let us first show that

$$\rho(x + \lambda x, K) \geq \lambda \delta \quad \text{for all } \lambda > 0, \ x \in \partial K. \tag{22}$$

Evidently (22) is satisfied if

$$\{x + \lambda(x + z) : \lambda > 0, |z| < \delta\} \cap K = \emptyset \ \text{ for all } x \in \partial K.$$

Given $x \in \partial K$, Mazur's theorem yields $x^* \in X^*$ with $x^* \neq 0$ and $x^*(y) \leq \gamma := x^*(x)$ for all $y \in K$, since K is convex with nonempty interior. Of course $\overline{B}_\delta(0) \subset K$ implies $|x^*(z)| < \gamma$ for $|z| < \delta$, hence

$$x^*(x + \lambda(x+z)) = (1+\lambda)x^*(x) + \lambda x^*(z) > \gamma$$

and therefore $x + \lambda(x+z) \notin K$ if $\lambda > 0$ and $|z| < \delta$, i.e. (22) holds. By means of (22) we obtain

$$\rho(v(\tau) + h_n(v'_+(\tau) + \epsilon v(\tau) + \tilde{e}_n), K) =$$

$$\rho((1+\epsilon h_n)(v(\tau) + h_n(v'_+(\tau) + e_n)) + h_n\hat{e}_n, K) \geq \epsilon h_n \delta - h_n|\hat{e}_n|$$

with $\hat{e}_n = e_n - \tilde{e}_n - h_n\epsilon(v'_+(\tau) + \tilde{e}_n) \to 0$, hence the contradiction

$$v(\tau) + h_n(v'_+(\tau) + \epsilon v(\tau) + \tilde{e}_n) \notin K \text{ for all large } n \geq 1.$$

Therefore, every mild solution v of (19) with $v_0 \in D(A)$ satisfies $v(t) \in K$ on $[0,b]$.
From this fact, the claim obviously follows by approximation of u_0 by $(u_{0,n}) \subset K \cap D(A)$, given that $\overline{K \cap D(A)} = K_A$. To see that the latter equality holds, let $u_0 \in K_A$ and $\eta > 0$ be given. Choose $u_1 \in \overset{\circ}{K} \cap D(A)$ and let $u_h = (1-h)u_0 + hu_1 \in \overset{\circ}{K}$, where $h > 0$ is so small that $|u_0 - u_h| \leq \eta/2$. Then $u_{0,\eta} = J_\lambda u_h$ satisfies $u_{0,\eta} \in K \cap D(A)$ and $|u_0 - u_{0,\eta}| \leq \eta$ if $\lambda > 0$ is sufficiently small. □

By means of Lemma 5.3 we obtain

Theorem 5.4 *Let X be a real Banach space such that X and X^* are uniformly convex. Let A be m-accretive in X such that $-A$ generates a compact semigroup, $K \subset X$ be closed bounded convex with $\overset{\circ}{K} \cap D(A) \neq \emptyset$. Suppose that $f : \mathbb{R}_+ \times K_A \to X$ is Carathéodory and T-periodic such that (10) and (17) hold. Then the evolution problem (8) has a T-periodic mild solution.*

Proof. The different subtangential condition requires another reduction to separable X. Fix $\tau \in J$ such that $f(\tau, \cdot)$ is continuous and bounded, consider the initial value problem

$$u' + Au \ni f(\tau, u) \text{ on } [0,1], \quad u(0) = u_0 \tag{23}$$

and let

$$D_n = \{u(1/n; u_0) : u_0 \in K_A, u(\cdot; u_0) \text{ is a mild solution of (23)}\}.$$

Due to compactness of $S(t)K_A$ for $t > 0$ and boundedness of K and $f(\tau, \cdot)$ it holds that $D_n \subset K_A$ is relatively compact for all $n \geq 1$. Let M_n be a countable dense subset of D_n and let $M = \bigcup_{n \geq 1} M_n$. Then $\overline{M} = K_A$. Indeed, given $\epsilon > 0$ and $u_0 \in K_A$, there exists a mild solution u of (23) by Theorem 4.5. Evidently $|u(1/n) - u_0| \leq \epsilon/2$ for some large n, hence there is $x_n \in M_n \subset M$ with $|x_n - u_0| \leq \epsilon$.

Now choose $N \subset J$ of measure zero such that all $f(t,\cdot)$ are continuous for $t \in J_0 := J \setminus N$, and $\overline{\text{span}} f(J_0 \times \{x\})$ is separable for all $x \in M$. Then

$$X_0 = \overline{\text{span}}\Big(f(J_0 \times M) \cup K_A\Big)$$

is a closed separable subspace of X. Since we may redefine f by $f(t,x) := 0$ on $N \times K_A$ without changing the solution sets of the initial value problems corresponding to (8), it follows that w.l.o.g. all $f(t,\cdot)$ are continuous. This implies $f : J \times K_A \to X_0$, hence f is almost continuous by Lemma 3.4. It therefore suffices to consider jointly continuous and bounded f, since reduction to this case is now possible as in step 2 of the proof of Theorem 5.3.

We may assume $\overline{B}_\delta(0) \subset K$ for some $\delta > 0$. Fix $\epsilon > 0$, let $g_\epsilon : J \times K_A \to X$ be locally Lipschitz such that $|f - g_\epsilon|_\infty \le \frac{1}{2}\epsilon\delta$ and define f_ϵ by $f_\epsilon(t,x) = g_\epsilon(t,x) - \epsilon x$ on $J \times K_A$. Then (17) holds for f_ϵ instead of f, which follows by the arguments given behind (18), hence $f_\epsilon(t,x) \in T_K^A(x)$ on $[0,T) \times K_A$ by Lemma 5.3. By the proof of Theorem 5.3 it is now obvious that (14) has a mild solution u_ϵ and consideration of $\epsilon \to 0+$ yields a T-periodic mild solution of (8). $\square$

In the autonomous case, existence of a stationary solution is again a direct consequence of Theorem 5.4.

Corollary 5.2 *Let X be a real Banach space such that X and X^* are uniformly convex. Let A be m-accretive in X such that $-A$ generates a compact semigroup, $K \subset X$ be closed bounded convex with $\mathring{K} \cap D(A) \neq \emptyset$ and $f : K_A \to X$ be continuous and bounded such that $f(x) \in T_K(x)$ on K_A. Then there exists $x \in D(A) \cap K$ such that $f(x) \in Ax$.*

5.3 Sums of accretive operators

Let A be m-accretive in a real Banach space X. In applications one often has to deal with operators of the type $A + F$, hence it is important to have sufficient conditions guaranteeing that this sum is m-accretive again. Such criteria are well known in two different settings, namely in uniformly smooth Banach spaces or for continuous F; see Remark 5.6 below for more details. The purpose of this section is to obtain sufficient conditions for multivalued F of usc type. For this purpose, the following characterization of m-accretivity will be needed.

Lemma 5.4 *Let A be an accretive operator in a real Banach space X. Then A is m-accretive if and only if $gr(A)$ is closed and*

$$\underline{\lim}_{h \to 0+} h^{-1}\rho(x + hz, R(I + hA)) = 0 \quad \text{for all } x \in \overline{D(A)} \text{ and all } z \in X. \tag{24}$$

This is Theorem 5.2 in Kobayashi [73].

Theorem 5.5 *Let A be m-accretive in a real Banach space X and $F : \overline{D(A)} \to 2^X \setminus \emptyset$ be usc with compact convex values such that $A + F$ is accretive. Then $A + F$ is m-accretive.*

Proof. Let $B = A+F$ with $D(B) := D(A)$. Then B has closed graph, since $(x_n, y_n) \in \mathrm{gr}\,(B)$ means $y_n = u_n + v_n$ with $u_n \in Ax_n$ and $v_n \in F(x_n)$, hence $(x_n, y_n) \to (x, y)$ implies $v_n \in F(x) + B_\epsilon(0)$ for all $n \geq n_\epsilon$ and therefore $v_{n_k} \to v \in F(x)$ for some subsequence (v_{n_k}) of (v_n), hence also $u_{n_k} \to u := y - v$ and $u \in Ax$ by closedness of $\mathrm{gr}\,(A)$.

In order to establish (24) we may assume $z = 0$, since for any $z \in X$ the map F_z, defined by $F_z(x) := F(x) - \{z\}$ on $\overline{D(A)}$, has the same properties as F. So we are done by Lemma 5.4, if

$$\underline{\lim}_{h\to 0+} h^{-1}\rho(x, R(I + hB)) = 0 \qquad \text{on } \overline{D(B)}. \tag{25}$$

Fix $x \in \overline{D(B)}$, let $h > 0$, $C = F(x)$ and $G(z) = F(J_h(x - hz))$ for $z \in X$ where $J_h = (I + hA)^{-1}$. Evidently, G is usc with compact convex values. Hence, given $\epsilon > 0$, there exists a continuous $g_\epsilon : C \to X$ such that $g_\epsilon(z) \in G(B_\epsilon(z) \cap C) + B_\epsilon(0)$ on C, by Proposition 2.2. Let $G_\epsilon(z) = P_C(g_\epsilon(z))$ for $z \in C$, where $P_C(\cdot)$ denotes the metric projection onto C. Then $G_\epsilon : C \to 2^C \setminus \emptyset$ is also usc with compact convex values, since P_C has this properties. Therefore G_ϵ has a fixed point $z_\epsilon \in C$ by Lemma 2.1. Given $h_n \searrow 0$ and $\epsilon_n \searrow 0$ we repeat the previous arguments to obtain fixed points z_n of the corresponding G_{ϵ_n}, i.e. we get a sequence $(z_n) \subset C$ such that

$$z_n \in P_C(y_n) \qquad \text{and} \qquad y_n \in F(J_{h_n}(x - h_n(B_{\epsilon_n}(z_n) \cap C))) + B_{\epsilon_n}(0).$$

In particular, there are $e_n, \hat{e}_n \in B_{\epsilon_n}(0)$ such that

$$y_n - e_n \in F(J_{h_n}(x - h_n\hat{z}_n)) \quad \text{ with } \hat{z}_n = z_n + \hat{e}_n \in C. \tag{26}$$

Now $x_n := J_{h_n}(x - h_n\hat{z}_n)$ satisfies $|x_n - x| \leq h_n|\hat{z}_n| + |J_{h_n}x - x|$, i.e. $x_n \to x$ as $n \to \infty$. We may therefore assume $y_n \to y$ for some $y \in F(x)$. Without loss of generality we also have $z_n \to z$ for some $z \in C$, $z_n \in P_C(y_n)$ implies $z \in P_C(y)$, hence $P_C(y) = \{y\}$ yields $y_n - z_n \to 0$. Together with (26) this means $\hat{z}_n \in F(x_n) + \tilde{e}_n$ for some $\tilde{e}_n \to 0$. Now recall that $x - h_n\hat{z}_n = J_{h_n}(x - h_n\hat{z}_n) + h_nA_{h_n}(x - h_n\hat{z}_n)$, where A_{h_n} denotes the Yosida approximation of A. We therefore obtain

$$x \in x_n + h_n(Ax_n + F(x_n) + \tilde{e}_n),$$

i.e. (25) holds. $\square$

From the viewpoint of applications, the assumptions on F in Theorem 5.5 are very strong, since the values will often be only weakly compact and convex. As a typical example consider $X = L^1(\Omega)$ and $F(u) = \{v \in L^1(\Omega) : v(x) \in \beta(u(x)) \text{ a.e. on } \Omega\}$ where β is a maximal monotone graph in $\mathbb{R}$. Here F is weakly usc with weakly compact convex values if β is bounded, say. In such a situation the proof of Theorem 5.5 breaks down, and we only have the following partial result.

Proposition 5.1 *Let A be m-accretive in a real Banach space X and demiclosed with compact resolvents. Let $F : \overline{D(A)} \to 2^X \setminus \emptyset$ be weakly usc and bounded with weakly compact convex values such that $A + F$ is accretive. Then $A + F$ is m-accretive.*

Proof. To obtain $R(I+A+F) = X$ it suffices to solve $x + Ax + F(x) \ni 0$, since $\tilde{F} = F - \{y\}$ has the same properties as F for every $y \in X$. Now $x \in D(A)$ is a solution of $x+Ax+F(x) \ni 0$ iff $y + F(x) \ni 0$ with $x = J_1 y$, hence we let $G(y) = -F(J_1 y)$ and look for a fixed point of G. Since F is bounded there is a ball $B = \overline{B}_R(0)$ such that $G(B) \subset B$, and $G(B)$ is weakly relatively compact by Proposition 2.3 and the fact that $J_1(B)$ is relatively compact. Let $K = \overline{\mathrm{conv}} G(B)$ and consider $G : K \to 2^K \setminus \emptyset$. We claim that G is usc with respect to the weak topology, i.e. $G^{-1}(C)$ is weakly closed whenever $C \subset X$ is weakly closed. Given such C it suffices to show that $G^{-1}(C)$ is weakly sequentially closed, since $G^{-1}(C) \subset K$ is weakly relatively compact and hence its weak closure coincides with its weak sequential closure. Given $(y_n) \subset G^{-1}(C)$ with $y_n \rightharpoonup y$, let $z_n \in G(y_n) \cap C$ and $x_n = J_1 y_n$, where we may assume $x_n \to x = J_1 y$ since $(J_1 y_n)$ is relatively compact and A is demiclosed. Then Proposition 2.3 yields $z_{n_k} \rightharpoonup z \in -F(x) \cap C$, i.e. $y \in G^{-1}(C)$.

Therefore, if X is equipped with the weak topology, then $K \subset X$ is compact convex and $G : K \to 2^K \setminus \emptyset$ is usc with closed convex values. Hence G has a fixed point by the multivalued version of Tychonov's theorem (see e.g. Theorem 9.B in Zeidler [116]). $\square$

5.4 Remarks

Remark 5.1 Theorem 5.1 is taken from Bothe [23]. By inspection of the proof it is rather obvious that the solution set $\mathcal{U}(u_0)$ of (1) is also a compact R_δ-set in the situations as described in Theorem 3.1 and Theorem 3.5. Part (b) of the result corresponding to Theorem 3.1 is essentially Theorem 3.3 in Tolstonogov/Umanskii [105] (where $F(t,\cdot)$ is supposed to be usc), and the fact that $\mathcal{U}(u_0)$ is a compact R_δ-set under the assumptions of Theorem 3.5 improves Theorem 3.5 of the same paper where, in particular, the much stronger compactness condition $\beta(F(t,B)) = 0$ for $t \in J$ and all bounded $B \subset X$ is imposed. In the situation of Theorem 3.1, the following simplification in the proof of the corresponding version of Theorem 5.1 is possible: since (2) yields a priori bounds for all $\mathcal{U}_n$ and the semigroup generated by $-A$ is compact, all $\overline{\mathcal{U}}_n$ are compact by Lemma 3.1, hence Lemma 5.1 is not needed then.

Theorem 5.2 remains valid if the semigroup generated by $-A$ is only equicontinuous, given that F satisfies the additional compactness condition

$$\lim_{h\to 0+} \beta(F(J_{t,h} \times B)) \le k(t)\beta(B) \quad \text{with } k \in L^1(J) \text{ for all bounded } B \subset K,$$

where $J_{t,h}$ denotes $[t-h, t+h] \cap J$; observe that application of Lemma 3.7 together with the latter condition yields $\beta_0(\mathcal{U}_n) \to 0+$. In the special case of linear A, Theorem 5.2 as well as the just mentioned variant thereof are contained in Bader/Kryszowski [10].
More information and additional references related to the special case $A = 0$ can be found in §7 and §9.3 in Deimling [42].

Remark 5.2 In Vrabie [113] existence of a T-periodic solution of (8) is obtained in the following situation: X is a real Banach space, A is an operator in X with $\overline{D(A)}$ convex such that $A - \omega I$ is m-accretive for some $\omega > 0$ and $-A$ generates a compact semigroup, $f : \mathbb{R}_+ \times \overline{D(A)} \to X$ is T-periodic and Carathéodory satisfying

$$\lim_{R\to\infty} \frac{1}{R} \sup\{|f(t,x)| : t \geq 0, x \in \overline{D(A)} \cap \overline{B}_R(0)\} < \omega. \tag{27}$$

The following observations clearify the relation to Theorem 5.3. One may assume $0 \in D(A)$ and $0 \in A(0)$ after a shift, and this does not affect (27). Extend f to a Carathéodory function on all of $\mathbb{R}_+ \times X$ by means of $f(t,x) := f(t, J_{\lambda(x)}x)$ for $x \notin \overline{D(A)}$ where $\lambda(x) = \rho(x, \overline{D(A)})$, and notice that (27) together with $|J_\lambda x| \leq |x|$ implies $|f(t,x)| < \omega R$ on $\mathbb{R}_+ \times \overline{B}_R(0)$ if $R > 0$ is sufficiently large. Hence $f_\omega := f - \omega I$ satisfies $f_\omega(t,x) \in T_K(x)$ on $[0,T) \times K$ with $K = \overline{B}_R(0)$. Since $A_\omega := A - \omega I$ is m-accretive with compact semigroup and $(I + \lambda A_\omega)^{-1} K \subset K$ for all $\lambda > 0$, application of Theorem 5.3 to A_ω, f_ω and $K = \overline{B}_R(0)$ yields a T-periodic mild solution u of

$$u' + (A - \omega I)u \ni f(t,u) - \omega u \quad \text{on } \mathbb{R}_+.$$

It is easy to see that u is also a mild solution of (8), hence the result mentioned above is a consequence of Theorem 5.3.

Remark 5.3 In the situation as described in Lemma 5.3 it is easy to see that (11) implies

$$\Big(f(t,x) - Ax\Big)^0 \in T_K(x) \quad \text{for all } t \in [0,T),\ x \in K \cap D(A), \tag{28}$$

hence (28) is a necessary condition for existence of mild solutions. Indeed, given $t_0 \in [0,T)$ and $x_0 \in K \cap D(A)$, the mild solution $u(\cdot)$ of

$$u' + Au \ni f(t_0, x_0) \ \text{ on } [0,1], \ \ u(0) = x_0$$

satisfies $h_n^{-1}\rho(u(h_n), K_A) \to 0$ for some sequence $h_n \to 0+$, by (11). On the other hand $u(\cdot)$ has a derivative $u_+'(\cdot)$ from the right and

$$u_+'(t) = \Big(f(t_0, x_0) - Au(t)\Big)^0 \quad \text{on } [0,1).$$

Hence $u(h_n) = x_0 + h_n u_+'(0) + h_n e_n$ with $e_n \to 0$ implies (28).

Remark 5.4 In the special case when X is a Hilbert space and $K = \overline{B}_R(0)$ for some $R > 0$, Theorem 5.4 includes Theorem 1.1 in Cascaval/Vrabie [33], where it is also assumed that f is jointly continuous. Section 4 of this paper contains an example in $L^2(\Omega)$ with $A = -\Delta_p$ that illustrates the advantage of condition (17) compared to the separated assumptions (12).

In Hirano [68] existence of T-periodic solutions for (8) is established under the assumptions that A is a subdifferential in a Hilbert space, $-A$ generates a compact semigroup, $f : \mathbb{R} \times X \to X$ is T-periodic and Carathéodory with $|f(t,x)| \leq c(1 + |x|)$ on $\mathbb{R} \times X$ such that $\langle x, f(t,x) - y\rangle \leq a - b|x|^2$ for all $t \in \mathbb{R}$, $x \in D(A)$ and $y \in Ax$ with $a, b > 0$. This is again a special case of Theorem 5.4, since the latter inequality implies (17) with $K = \overline{B}_R(0)$ for all large $R > 0$.

Remark 5.5 Existence of T-periodic solutions for (8) with compact f is a delicate problem. In the semilinear case, i.e. for m-accretive, linear and densely defined A, it is known that compactness of the semigroup $S(t)$ can be replaced by compactness of f, under the additional assumption that 1 belongs to the resolvent set $\rho(S(T))$. This is contained in Theorem 3 in Prüss [96], which also includes the case of a compact semigroup; the other conditions are: $f : \mathbb{R}_+ \times K \to X$ continuous, bounded and T-periodic where $K \subset X$ is closed bounded convex with nonempty interior, and f satisfies the necessary subtangential condition. The extra assumption "$1 \in \rho(S(T))$" cannot be dropped since Example 1 in Deimling [40] provides a compact $f : l^2 \to l^2$ satisfying $\langle f(x), x\rangle < 0$ for all $x \in l^2$ with $|x|_2 = r$ for a given $r > 0$, such that $u' = f(u)$ has no T-periodic solution for arbitrary $T > 0$.
In the semilinear setting, extensions of the above mentioned result to the case of multivalued usc perturbations are given in Bader [9] and in Bader/Kryszewski [10].

Remark 5.6 Theorem 5.5 is Theorem 1 in Bothe [20]. Specialized to the case of single-valued perturbations, the conditions on F become "$F : \overline{D(A)} \to X$ continuous such that $A + F$ is accretive". In this situation the result is known and, using Lemma 5.4, it was first proved in Kobayashi [73] where it is Theorem 5.3. Independently, the same result was obtained in Pierre [94] by means of locally Lipschitz approximations of F.

The first result about continuous perturbations of m-accretive operators is Theorem 1 in Barbu [13], where the assumptions on F are $F : X \to X$ continuous and accretive. In this situation F is in fact s-accretive, hence $A+F$ is automatically accretive if A has this property. This is not true for multivalued F, as shown by the following

Example 5.2 Let $X = \mathbb{R}^2$ with $|\cdot|_1$ and $A : D(A) \to 2^X \setminus \emptyset$ be given by $Ax = \{(s,s) : s \in \mathbb{R}\}$ on $D(A) = \{0\} \times \mathbb{R}$. Evidently $R(I + \lambda A) = X$ for all $\lambda > 0$. Moreover A is accretive, since $x, \overline{x} \in D(A)$, $y \in Ax$, $\overline{y} \in A\overline{x}$ implies $y - \overline{y} = (s - \overline{s}, s - \overline{s})$ with $s, \overline{s} \in \mathbb{R}$, hence

$$[x - \overline{x}, y - \overline{y}] \geq |s - \overline{s}| + (s - \overline{s}) \operatorname{sgn}(x_2 - \overline{x}_2) \geq 0.$$

Let $F : X \to 2^X \setminus \emptyset$ be defined by

$$F(x) = \begin{cases} \{(-1,0)\} & \text{if } x_1 < 0 \\ [-1,1] \times \{0\} & \text{if } x_1 = 0. \\ \{(1,0)\} & \text{if } x_1 > 0 \end{cases}$$

Observe that F is of the same type as the operator A considered in part (a) of Example 3.1 in §3.1, hence F is accretive. In addition, F is obviously usc with compact convex values. Now $A + F \equiv \{(s,t) \in \mathbb{R}^2 : |s-t| \leq 1\}$ on $D(A)$ which is not accretive, since e.g. $x = (0,1)$, $\overline{x} = (0,0)$, $y = -x$, $\overline{y} = \overline{x}$ yield $[x - \overline{x}, y - \overline{y}] = [x, -x] = -|x|_1 = -1$. ◇

If both A and B are m-accretive operators with $D(A) \cap D(B) \neq \emptyset$, other well-known criteria for m-accretivity of $A + B$ which apply if X^* is uniformly convex can be found, e.g. in Barbu [14] and Benilan/Crandall/Pazy [17]. Here, let us only mention Theorem 3 in Prüss [98], saying that $A + B$ is m-accretive given that A and B have this property, X and X^* are uniformly convex and $D(A) \cap \operatorname{int}(D(B)) \neq \emptyset$.

Chapter 3

Applications

Consider a reaction-diffusion system of the form

$$\begin{aligned}
&\frac{\partial u_k}{\partial t} = \Delta\varphi_k(u_k) + g_k(t, u_1, \ldots, u_m) && \text{in } (0,\infty)\times\Omega \\
&\frac{\partial \varphi_k(u_k)}{\partial \nu} = 0 && \text{on } (0,\infty)\times\Gamma \\
&u_k(0,\cdot) = u_{0,k} && \text{in } \Omega
\end{aligned} \tag{3.1}$$

for $k = 1, \ldots, m$ where the components u_k denote the concentration of certain chemical species, the functions φ_k are continuous and strictly increasing with $\varphi_k(0) = 0$, $g : \mathbb{R}_+^m \to \mathbb{R}^m$ is continuous and $\Omega \subset \mathbb{R}^n$ is open bounded with sufficiently smooth boundary Γ.

System (3.1) admits an abstract formulation in $X = L^1(\Omega)^m$, given as

$$u' + Au \ni f(t,u) \ \text{ on } \mathbb{R}_+, \ \ u(0) = u_0 \tag{3.2}$$

where A corresponds to nonlinear diffusion in (3.1), and the nonlinear reacting force $f : D(f) \subset \mathbb{R}_+ \times X \to X$ is defined by $f(t,u)(x) = g(t, u(x))$ for $x \in \Omega$. Due to the assumptions on φ_k mentioned above, the operator A is m-accretive and T-accretive in X, and the semigroup generated by $-A$ satisfies additional compactness properties. This follows from known facts about nonlinear diffusion of type $\Delta\varphi(v)$ which are collected in §6.1. Hence (3.1) leads to an evolution problem that falls into the scope of the abstract theory given in Chapter 2.

Due to the physical background additional constraints appear naturally: nonnegativity of the solutions of (3.1) will always be a minimal requirement, and in the abstract formulation f will usually be defined on certain subsets of $\mathbb{R}_+ \times X$, only. Consequently, invariance and viability results are useful for the study of problems like (3.1). This is the subject of §6.2 where we show how the abstract theory applies to carry over some common invariance techniques that are well-known in the semilinear setting, like invariant or contracting rectangle, to such reaction-diffusion systems with nonlinear diffusion.

In §6.3 we study the following more realistic model that describes a concrete process encountered in chemical engineering in the context of heterogeneous catalysis.

$$\begin{aligned}
\frac{\partial u_k}{\partial t} &= \Delta\varphi_k(u_k) + r_k(t,x,u_1,\dots,u_m) && \text{for } t>0,\ x\in\Omega \\
\frac{\partial \varphi_k(u_k)}{\partial \nu} &= \gamma_k(c_k - h_k(u_k)) && \text{for } t>0,\ x\in\Gamma \\
\frac{dc_k}{dt} &= -\int_\Gamma \gamma_k(c_k-h_k(u_k))d\sigma + R_k(t,c_1,\dots,c_m) && \text{for } t>0 \\
k &= 1,\dots,m.
\end{aligned} \tag{3.3}$$

Here, let us just mention some particular features of the underlying process. Certain chemical reactions are performed inside porous catalytic pellets that are suspended in a surrounding bulk phase, where the latter is assumed to be ideally mixed. Therefore, in addition to nonlinear diffusion and reaction inside the pellets, macroscopic convection and reaction in the bulk phase as well as interfacial mass transport have to be taken into account. Hence (3.3) describes a two-phase process where u_k denotes the concentration of a chemical species inside the pellets, while c_k represents the concentration of the same species in the bulk phase. In practice such a process is sometimes operated with periodically varying feeds in order to increase the performance with respect to conversion or selectivity, hence the question of existence of periodic solutions appears naturally.

Based especially on compactness properties of the operator

$$A\begin{pmatrix} v \\ c \end{pmatrix} = \begin{pmatrix} -\Delta\varphi(v) \\ \int_\Gamma \gamma(c-h(v))d\sigma \end{pmatrix} \text{ in } X = L^1(\Omega)\times\mathbb{R} \text{ with}$$

$$D(A) = \{(v,c)\in X : \varphi(v)\in W^{1,1}(\Omega), \Delta\varphi(v)\in L^1(\Omega), \frac{\partial\varphi(v)}{\partial\nu} = \gamma(c-h(v)) \text{ on } \Gamma\},$$

we show that (3.3) admits an abstract formulation of type (3.2) to which the abstract theory from Chapter 2 applies. In particular, we establish existence of T-periodic and stationary solutions under fairly realistic assumptions.

A different aspect of chemically reacting systems is studied in §7 and §8. Given a system of chemical reactions it often occurs that some of the reactions take place at a considerably higher rate than the remaining ones, in particular if radical or ionic reactions are involved which run under enormous speed. Then a natural question is whether solutions to corresponding models converge to the solution of an associated limit problem if the rate constants of all fast reactions tend to infinity. To investigate this question we distinguish between fast reversible and fast irreversible reactions, since different techniques are needed. In both situations an important first step is to study the ideally mixed case with macroscopic convection which leads to initial value problems for ordinary differential equations of type

$$\dot c = f(c) + kNR(c) \ \text{ on } \mathbb{R}_+,\ \ c(0)=c_0 \tag{3.4}$$

with a large parameter $k > 0$ corresponding to the large rate constants; here f represents feeds plus additional slow reaction, N is the stoichiometric matrix and $R = (R_1, \ldots, R_m)$ is the vector of rate functions.

In §7.1 we consider initial value problem (3.4) in the particular case of two fast irreversible reactions and obtain convergence of solutions, as k tends to infinity, to the solution of a discontinuous limit problem. If the fast irreversible reactions take place in the bulk phase of the two-phase process mentioned above, then the limit problem is a reaction-diffusion system of type (3.3) but with discontinuous nonlinearities R_k.

The case of fast reversible reactions is studied in §8, where we use Lyapunov functions techniques to establish convergence of the solutions of (3.4) for a general system of independent fast reactions.

The passage to infinite reaction speed becomes more difficult if diffusion is taken into account, and here we only consider the case of a single reaction $A + B \to P$, repectively $A + B \rightleftharpoons P$. If the reaction takes place inside an isolated vessel this leads to the model problem

$$\begin{aligned}
\frac{\partial c_A}{\partial t} &= D_A \Delta c_A - k(c_A c_B - \kappa c_P) \ \text{ in } \Omega, \qquad \frac{\partial c_A}{\partial \nu} = 0 \ \text{ on } \partial\Omega \\
\frac{\partial c_B}{\partial t} &= D_B \Delta c_B - k(c_A c_B - \kappa c_P) \ \text{ in } \Omega, \qquad \frac{\partial c_B}{\partial \nu} = 0 \ \text{ on } \partial\Omega \\
\frac{\partial c_P}{\partial t} &= D_P \Delta c_P + k(c_A c_B - \kappa c_P) \ \text{ in } \Omega, \qquad \frac{\partial c_P}{\partial \nu} = 0 \ \text{ on } \partial\Omega
\end{aligned} \tag{3.5}$$

with $D_j > 0$ and $\kappa \geq 0$. In the irreversible case $\kappa = 0$ we allow for nonlinear diffusion and more general reaction kinetics in (3.5), and obtain convergence of the solutions as $k \to \infty$ to the solution of a free boundary problem by means of nonlinear semigroup theory; this is the subject of §7.2. The reversible case $\kappa > 0$ is studied in §8.3 where we solve the corresponding singular limit problem under the strong extra assumptions of equal diffusion coefficients.

§6 Reaction-Diffusion Systems with Nonlinear Diffusion

We consider certain classes of reaction-diffusion systems with nonlinear diffusion, say

$$u_t = \Delta\Phi(u) + g(u) \text{ in } (0,\infty)\times\Omega,\ \ \Phi(u) = 0 \text{ on } (0,\infty)\times\partial\Omega,\ \ u(0,\cdot) = u_0 \text{ in } \Omega$$

in the simplest case; here $u = (u_1, \dots, u_m)$ and $\Delta\Phi(u) = (\Delta\varphi_1(u_1), \dots, \varphi_m(u_m))$ with continuous, strictly increasing φ_k. The purpose of the present paragraph is to show how the abstract results of Chapter 2 can be applied in order to obtain qualitative information about the solutions of such systems.

In practice, chemical reactions are often performed inside catalytic pellets of high porosity. Therefore we consider model problems with nonlinear diffusion of the type $\Delta\varphi(u)$ since the latter includes the common models for diffusion in porous media as a special case; see e.g. Vazquez [107] for more information concerning the porous medium equation.

6.1 Nonlinear diffusion of type $\Delta\varphi(u)$

We collect several known facts concerning the abstract formulation of the scalar nonlinear diffusion equation

$$u_t = \Delta\varphi(u) \ \text{ in } (0,T)\times\Omega, \quad -\frac{\partial\varphi(u)}{\partial\nu} \in \beta(u) \text{ on } (0,T)\times\Gamma, \quad u(0,\cdot) = u_0 \text{ in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is open bounded with sufficiently smooth boundary Γ, $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing with $\varphi(0) = 0$ and β is a maximal monotone graph in $\mathbb{R}$ with $0 \in \beta(0)$. Define the operator A in $L^1(\Omega)$ by means of

$$\left.\begin{array}{c} Au = -\Delta\varphi(u) \quad \text{for } u \in D(A), \text{ where} \\ D(A) = \{u \in L^1(\Omega) : \varphi(u) \in W^{1,1}(\Omega), \Delta\varphi(u) \in L^1(\Omega), -\dfrac{\partial\varphi(u)}{\partial\nu} \in \beta(u) \text{ on } \Gamma\}. \end{array}\right\} \tag{2}$$

Here $\Delta\varphi(u)$ is to be understood in the sense of distributions. More precisely, definition (2) is an abbreviation of the following exact formulation:

$$(u,w) \in \mathrm{gr}\,(A) \ \text{ iff there exists } g \in L^1(\Gamma) \text{ such that } -g(x) \in \beta(u(x)) \text{ a.e. on } \Gamma$$

$$\text{and } v = \varphi(u) \text{ is the weak solution of } \ -\Delta v = w \ \text{ in } \Omega, \quad \frac{\partial v}{\partial\nu} = g \ \text{ on } \Gamma,$$

i.e. $v \in W^{1,1}(\Omega)$ is such that

$$\int_\Omega \langle\nabla u, \nabla f\rangle dx = \int_\Omega wf\,dx + \int_\Gamma gf\,d\sigma \quad \text{for all } f \in C^1(\overline{\Omega});$$

here $u_{|\Gamma}$ is understood as $\varphi^{-1}(\varphi(u)_{|\Gamma})$.

Lemma 6.1 *Let $\Omega \subset \mathbb{R}^n$ be open bounded with C^2-boundary Γ, $\varphi : \mathbb{R} \to \mathbb{R}$ continuous, strictly increasing with $\varphi(0) = 0$ and β a maximal monotone graph in $\mathbb{R}$ with $0 \in \beta(0)$. Let $A : D(A) \subset L^1(\Omega) \to L^1(\Omega)$ be defined by (2). Then A is m-accretive and T-accretive with $\overline{D(A)} = L^1(\Omega)$. In addition, $|(J_\lambda u)^\pm|_p \leq |u^\pm|_p$ for all $\lambda > 0$, $p \in [1,\infty]$ and $u \in L^p(\Omega)$.*

This is Theorem II.2.1 and Corollaire II.2.2 in Benilan [15]. Use of $L^1(\Omega)$ as the phase space is natural because accretivity can be obtained only in $L^1(\Omega)$-norm (see the proof of Lemma 6.4 below), and physically the $L^1(\Omega)$-norm reflects conservation of mass.

To be able to apply the abstract results of Chapter 2, we need additional compactness properties of the semigroup $S(t)$ generated by $-A$. The subsequent results refer to the case of homogeneous Dirichlet boundary conditions, i.e. $\varphi(u) = 0$ on $(0,T) \times \Gamma$ in (1); notice that the latter corresponds to the choice $\beta = \{0\} \times \mathbb{R}$. In the situation of Lemma 6.1 (with $\beta = \{0\} \times \mathbb{R}$), the semigroup need not be compact, but compactness of $S(t)$ is guaranteed if, in addition, φ is continuously differentiable on $\mathbb{R} \setminus \{0\}$ such that

$$\varphi'(r) \geq c|r|^{\gamma-1} \text{ on } \mathbb{R} \setminus \{0\} \text{ with some } c > 0 \text{ and } \gamma > \max\{0, \frac{n-2}{n}\};$$

see e.g. Lemma 2.7.2 in Vrabie [112].

In the special case $\varphi(r) = |r|^{\gamma-1}r$, i.e. in case of the porous medium equation, above condition is optimal in the sense that the semigroup is not compact for $0 < \gamma < \frac{n-2}{n}$. This is a consequence of Theorem 8 in Brezis/Friedman [31]; see Remark 11 there.

On the other hand, the fact that φ is *strictly* increasing already implies compactness properties of the semigroup that are sufficient for our purpose. To state these properties, let $\mathcal{S}w$ denote the unique mild solution of

$$u' + Au = w(t) \text{ on } J = [0,a], \quad u(0) = u_0,$$

where $u_0 \in L^1(\Omega)$ is fixed, A is given by (2) and $w \in L^1(J; L^1(\Omega))$.

Lemma 6.2 *Let $\Omega \subset \mathbb{R}^n$ be open bounded with C^2-boundary Γ and $\varphi : \mathbb{R} \to \mathbb{R}$ continuous, strictly increasing with $\varphi(0) = 0$. Let A be defined by (2) with $\beta = \{0\} \times \mathbb{R}$.*

(a) Then $\{\mathcal{S}w : w \in W\}$ is relatively compact in $C(J; L^1(\Omega))$ if $W \subset L^1(J; L^1(\Omega))$ is weakly relatively compact.

(b) Let $u_0 \in L^\infty(\Omega)$ and (w_k) be bounded in $L^\infty(J; L^\infty(\Omega))$. Then $w_k \rightharpoonup w$ in $L^1(J; L^1(\Omega))$ implies $\mathcal{S}w_k \to \mathcal{S}w$ in $C(J; L^1(\Omega))$.

The first part is Théoreme 1 in Diaz/Vrabie [45], while the second one is Corollary 3.1 in Diaz/Vrabie [46]. To obtain this result, the main point is to show that $S(t)B$ is relatively compact in $L^1(\Omega)$ for $t > 0$ and B bounded in $L^\infty(\Omega)$. Formally, this property of $S(t)$ follows easily by multiplication of $u_t = \Delta\varphi(u)$ in (1) by $\varphi(u)$, respectively by $\varphi'(u)u_t$ and integration over $(s,t) \times \Omega$. This yields

$$|\Phi(u(t))|_1 + \int_0^t |\nabla\varphi(u(\tau))|_2^2 d\tau = |\Phi(u_0)|_1 \text{ for } t > 0 \quad \text{with } \Phi(r) = \int_0^r \varphi(\rho)d\rho,$$

respectively

$$|\nabla\varphi(u(t))|_2^2 \le |\nabla\varphi(u(s))|_2^2 \ \text{ for } 0 \le s \le t.$$

Therefore

$$|\nabla\varphi(u(t))|_2^2 \le \frac{1}{t}|\Phi(u_0)|_1 \ \text{ for } t > 0,$$

hence $\{S(t)u_0 : u_0 \in B\}$ is bounded in $W^{1,2}(\Omega)$ for all $t > 0$, in particular relatively compact in $L^1(\Omega)$ by compact embedding.

This yields the first part in Lemma 6.2 as follows. It suffices to consider $u_0 \in L^\infty(\Omega)$ and also $W \subset L^\infty(J; L^\infty(\Omega))$, since $W \subset L^1(J \times \Omega)$ is weakly relatively compact (for bounded Ω) iff

$$\sup_{w\in W} |w - w\chi_{\{|w|\le R\}}|_1 \to 0 \ \text{ as } R \to \infty.$$

For such u_0 and W the sections $U(t) = \{(\mathcal{S}w)(t) : w \in W\}$ are bounded in $L^\infty(\Omega)$ and satisfy $U(t) \subset S(h)U(t-h) + h\overline{B}_M(0)$ for $0 \le t-h \le t$ with some $M > 0$. Hence $U(t)$ is relatively compact in $L^1(\Omega)$ for $t \in J$ and then equicontinuity of $\mathcal{S}(W)$ follows by the standard argument used in §3 and §4.

In the situation of part (b) let $u_k = \mathcal{S}w_k$, where we may assume $u_k \to u$ in $C(J; L^1(\Omega))$ by (a). It is known that u_k is the weak solution (in the sense of distributions) of

$$u_t = \Delta\varphi(u) + w_k \ \text{ in } (0,T)\times\Omega, \quad \varphi(u) = 0 \ \text{ on } (0,T)\times\Gamma, \quad u(0,\cdot) = u_0 \text{ in } \Omega \tag{3}$$

and that weak solutions of (3) are unique; see Brezis/Crandall [30]. By the estimates given for $|\nabla\varphi(u)|_2$ above, we may also assume $\nabla\varphi(u_k) \rightharpoonup \nabla\varphi(u)$ in $L^2(J; L^2(\Omega))$, hence u is the unique weak solution of (3) with w instead of w_k, and therefore $u = \mathcal{S}w$. Here the crucial point is to have uniqueness of weak solutions of (3). The latter seems to be unknown under the general boundary condition considered in (1), but it holds under homogeneous Neumann conditions, given that $u_0 \in L^\infty(\Omega)$ and $w \in L^1(J; L^\infty(\Omega))$; see Benilan [15]. Therefore it is easy to check that Lemma 6.2 remains valid in case of homogeneous Neumann boundary conditions.

Let us finally mention that the formal arguments indicated above can be made precise by means of characterizations of Lyapunov functions; this will become clear in §6.3 below where we consider a more complicated situation.

6.2 Invariance techniques

In the present section we consider a class of reaction-diffusion systems with nonlinear diffusion and draw some consequences from the abstract theory about existence and viability of mild solutions. In the sequel we concentrate on the specific model problem

$$\begin{aligned}
&\frac{\partial u_k}{\partial t} = \Delta\varphi_k(u_k) + g_k(t,u) && \text{in } (0,\infty)\times\Omega \\
&\varphi_k(u_k) = 0 && \text{on } (0,\infty)\times\Gamma \qquad (k = 1,\dots,m) \\
&u_k(0,\cdot) = u_{0,k} && \text{in } \Omega,
\end{aligned} \tag{4}$$

where $\Omega \subset \mathbb{R}^n$ is open bounded with sufficiently smooth boundary Γ, $\varphi_k : \mathbb{R} \to \mathbb{R}$ continuous, strictly increasing and $g_k : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$ continuous. System (4) will always be considered as the abstract evolution problem

$$u_k' + A_k u_k = f_k(t,u) \quad \text{on } \mathbb{R}_+, \quad u_k(0) = u_{0,k} \quad (k = 1, \ldots, m) \tag{5}$$

in $L^1(\Omega)^m$, where $A_k u_k$ corresponds to $-\Delta\varphi_k(u_k)$ and the components of f are given by $f_k(t,u)(x) = g_k(t, u(x))$. Therefore, we call u a mild solution of (4) if u is a mild solution of the associated evolution problem (5).

By the proofs to follow it will be clear that the subsequent results are valid for other systems of type (5), given that the A_k are m-accretive in $L^1(\Omega)$, say, and enjoy the same properties as stated in Lemma 6.1 and 6.2.

In particular in the semilinear case, say problem (4) with $\varphi_k(r) = d_k r$ and $d_k > 0$, it is well known that invariance techniques can be useful to obtain qualitative properties of solutions like nonnegativity, global existence and asymptotic stability, existence of periodic solutions etc. In the simplest case, the basic idea within this approach is as follows. Suppose that C is a closed bounded subset of $\mathbb{R}^m$ which is weakly positively invariant for the ordinary differential equation

$$y' = g(t,y) \quad \text{on } \mathbb{R}_+$$

associated to (4), and consider $K = \{u \in L^1(\Omega)^m : u(x) \in C \text{ a.e. on } \Omega\}$. Then K is weakly positively invariant for $u_t = D\Delta u + g(t,u)$ with $D = \operatorname{diag}(d_1, \ldots, d_m)$, given that K is positively invariant for $u_t = D\Delta u$. Of course the latter condition implies severe restrictions on the structure of the sets C. In particular, in case of different diffusion coefficients ($d_i \neq d_k$ if $i \neq k$) it turns out that C has to be a "rectangle" such that $0 \in C$, i.e. $C = [a,b] \subset \mathbb{R}^m$ with $a_k \leq 0 \leq b_k$. Nevertheless, there are several applications from different areas where such invariant rectangles are available; examples can be found for instance in Chueh/Conley/Smoller [35], Hetzer/Schmidt [65], Smoller [103], Valencia [106] and the references given there.

This approach carries over to the model problem under consideration.

Theorem 6.1 *Let $\Omega \subset \mathbb{R}^n$ be open bounded with C^2-boundary Γ and $\varphi_k : \mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing with $\varphi_k(0) = 0$. Let $C = [a,b] \subset \mathbb{R}^m$ with $0 \in C$ and $g : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$ be continuous such that C is weakly positively invariant for $y' = g(t,y)$. Then $K = \{u \in L^1(\Omega)^m : u(x) \in C \text{ a.e. on } \Omega\}$ is weakly positively invariant for (4).*
In particular, (4) has a global mild solution for every $u_0 \in K$.

Proof. Let $X = L^1(\Omega)^m$ with $|u| = |u_1|_1 + \ldots + |u_m|_1$. For $k = 1, \ldots, m$ let the operator A_k be given by (2) with φ_k instead of φ and $\beta = \{0\} \times \mathbb{R}$, and define A by means of

$$Au = (A_1 u_1, \ldots, A_m u_m) \quad \text{on } D(A) = D(A_1) \times \ldots \times D(A_m).$$

Then the abstract formulation of (4) is given as the evolution problem

$$u' + Au = f(t,u) \text{ on } \mathbb{R}_+, \quad u(0) = u_0,$$

where $f : \mathbb{R}_+ \times K \to X$ is defined by $f(t,u)(x) = g(t,u(x))$ for $x \in \Omega$. Given $t_0 \geq 0$ and $u_0 \in K$, we have to show that the corresponding initial value problem has a mild solution on $[t_0, \infty)$, where it suffices to show that

$$u' + Au = f(t,u) \text{ on } J = [0,a], \quad u(0) = u_0 \tag{6}$$

has a mild solution for every choice of $a > 0$. Evidently f is continuous and bounded on $J \times K$, and $f(J \times K) \subset X$ is weakly relatively compact. Lemma 6.1 implies that A is m-accretive in X, and it follows from Lemma 6.2(a) that the solution operator $\mathcal{S} : L^1(J;X) \to C(J;X)$ of the quasi-autonomous problem associated with A maps weakly relatively compact sets into relatively compact sets. Therefore Theorem 4.2 applies and yields existence of a mild solution of (6), if

$$J_\lambda K \subset K \text{ for } \lambda > 0 \qquad \text{and} \qquad f(t,u) \in T_K(u) \text{ on } [0,a) \times K. \tag{7}$$

Let $K_k = \{v \in L^1(\Omega) : v(x) \in [a_k, b_k] \text{ a.e on } \Omega\}$ and $J_\lambda^k = (I + \lambda A_k)^{-1}$ for $\lambda > 0$. Given $v \in K_k$, Lemma 6.1 yields $|(J_\lambda^k v)^\pm|_\infty \leq |v^\pm|_\infty$, hence $J_\lambda^k v \in K_k$. Consequently, the first condition in (7) holds. To verify the second one, recall that g satisfies the necessary subtangential condition $g(t,y) \in T_C(y)$ on $\mathbb{R}_+ \times C$ since C is weakly positively invariant, and $z \in T_C(y)$ means $\lim_{h \to 0+} h^{-1} \rho(y + hz, C) = 0$ since C is closed convex. Then, given $u \in K$,

$$\frac{1}{h} \rho(u(x) + hg(t,u(x)), C) \to 0 \text{ as } h \to 0+ \text{ a.e. on } \Omega$$

together with the dominated convergence theorem implies $f(t,u) \in T_K(u)$ on $J \times K$; notice that

$$\rho(v,K) = \int_\Omega \rho(v(x), C)dx \text{ for every } v \in X,$$

which follows from the fact that $P_C(v(\cdot))$ has a measurable selection for every $v \in X$ by Lemma 2.2, where P_C denotes the metric projection onto C. $\square$

By the remarks given behind Lemma 6.2 it is obvious that Theorem 6.1 remains true if system (4) is replaced by

$$\begin{aligned}
&\frac{\partial u_k}{\partial t} = \Delta \varphi_k(u_k) + g_k(t,u) && \text{in } (0,\infty) \times \Omega \\
&\frac{\partial \varphi_k(u_k)}{\partial \nu} = 0 && \text{on } (0,\infty) \times \Gamma \qquad (k = 1, \dots, m) \\
&u_k(0,\cdot) = u_{0,k} && \text{in } \Omega.
\end{aligned} \tag{8}$$

In this case the condition $0 \in C$ can be dropped which gains more flexibility for finding invariant rectangles; notice that $J_\lambda^k K_k \subset K_k$ is still valid since J_λ^k is order-preserving (due to T-accretivity of A_k) and $J_\lambda^k v = v$ in case v is constant in Ω. Let us also mention that $g(t,y) \in T_C(y)$ with $C = [a,b]$ means:

$$y \in [a,b] \text{ with } y_k = a_k \text{ implies } g_k(t,y) \geq 0, \;\; y \in [a,b] \text{ with } y_k = b_k \text{ implies } g_k(t,y) \leq 0.$$

If the strict inequalities hold, i.e. if $g(t,y)$ belongs to the interior of $T_C(y)$ for every $(t,y) \in \mathbb{R}_+ \times C$, it is rather obvious that a solution of $y' = g(t,y)$ starting in C immediately enters smaller rectangles contained in C; in this case C is called a *contracting rectangle*. By means of this observation it is sometimes possible to obtain further qualitative information, like asymptotic stability, about the reaction-diffusion system under consideration if there is a nested family of such contracting rectangles. This is mainly of interest in the autonomous case $g : \mathbb{R}^m \to \mathbb{R}^m$, since it is then possible to check if such a family of contracting rectangles exists, merely by analysing the phase portrait of g. Several applications of this approach are given in §14 of Smoller [103]; see also Valencia [106] and the references given there.
By the next result, this technique is also at our disposal for systems of type (8).

Theorem 6.2 *Let $\Omega \subset \mathbb{R}^n$ be open bounded with C^2-boundary Γ, $\varphi_k : \mathbb{R} \to \mathbb{R}$ be continuous, strictly increasing with $\varphi_k(0) = 0$ and $g : \mathbb{R}^m \to \mathbb{R}^m$ be locally Lipschitz. Let $R(\tau) \subset \mathbb{R}^m$ for $\tau \in [0,1)$ be a family of contracting rectangles such that $R(\cdot)$ is continuous with respect to d_H and $R(\hat\tau) \subset R(\tau)$ for $\hat\tau > \tau$. Let $D(\tau) = \{u \in L^1(\Omega)^m : u(x) \in R(\tau) \text{ a.e on } \Omega\}$ for $\tau \in [0,1]$. Then, given $u_0 \in D(0)$, there is a unique mild solution u of (8) in $D(0)$, and for every $\tau \in [0,1)$ there is $t_0 = t_0(\tau, u_0) \geq 0$ such that $u(t) \in D(\tau)$ on $[t_0, \infty)$.*
If, in addition, $\bigcap_{\tau \in [0,1)} R(\tau) = \{y_\infty\}$ with some $y_\infty \in \mathbb{R}^m$, then $u_\infty \equiv y_\infty$ is a stationary solution of (8), and $|u_k(t) - u_{\infty,k}|_\infty \to 0$ as $t \to \infty$ for every mild solution u of (8) in $D(0)$.

Proof. Consider the abstract evolution problem (5) corresponding to (8), i.e. the components $A_k u_k$ refer to $-\Delta\varphi_k(u_k)$ with homogeneous Neumann boundary condition and $f : D(0) \to X$ is given by $f(u)(x) = g(u(x))$. Since g is Lipschitz continuous on the compact set $R(0)$ it follows that f is Lipschitz, and $g(y) \in T_{R(0)}(y)$ on $R(0)$ implies $f(u) \in T_{D(0)}(u)$ on $D(0)$. Due to Lemma 6.1 the operator A is m-accretive such that $J_\lambda D(0) \subset D(0)$ for $\lambda > 0$, hence Corollary 4.2 yields a unique mild solution u of (5) in $D(0)$. If the initial value belongs to $D(\tau)$ for some $\tau \in [0,1)$, then the same argument with $D(\tau)$ instead of $D(0)$ yields a mild solution of (5) in $D(\tau)$. Consequently, if the mild solution u in $D(0)$ satisfies $u(t_0) \in D(\tau)$ for some $t_0 \geq 0$ then $u(t) \in D(\tau)$ on $[t_0, \infty)$.

Let $\sigma(t) = \sup\{\tau \in [0,1) : u(t) \in D(\tau)\}$ for $t \geq 0$. Then $\sigma : \mathbb{R}_+ \to [0,1]$ is increasing due to the considerations above, hence $\sigma_\infty = \lim_{t\to\infty} \sigma(t)$ exists and the first assertion obviously holds if $\sigma_\infty = 1$. Suppose that $\sigma_\infty < 1$ and let $R = R(\sigma_\infty)$. Since $R = [a,b]$ is a contracting rectangle, g is continuous and R is compact, there is $\eta > 0$ such that

$$g_k(y) \geq 2\eta \text{ for } y \in [a,b] \text{ with } y_k = a_k, \quad g_k(y) \leq -2\eta \text{ for } y \in [a,b] \text{ with } y_k = b_k.$$

Hence there also is $\epsilon > 0$ such that

$$\begin{aligned} g_k(y) &\geq \eta \quad \text{for } y \in [a-\epsilon e, b+\epsilon e] \text{ with } |y_k - a_k| \leq \epsilon, \\ g_k(y) &\leq -\eta \quad \text{for } y \in [a-\epsilon e, b+\epsilon e] \text{ with } |y_k - b_k| \leq \epsilon, \end{aligned}$$

where $e = (1, \ldots, 1) \in \mathbb{R}^m$. We may also assume $2\epsilon \leq b_k - a_k$ for all k; notice that any contracting rectangle necessarily has nonempty interior. Let $J = [0, a]$ with $a = 2\epsilon/\eta$, define the tube $C : J \to 2^{\mathbf{R}^m} \setminus \emptyset$ by $C(t) = [a - \epsilon e + t\eta e, b + \epsilon e - t\eta e]$ and let

$$K(t) = \{u \in L^1(\Omega)^m : u(x) \in C(t) \text{ a.e. on } \Omega\} \quad \text{on } J.$$

Evidently $y \in C(t)$ for $t \in [0, a)$ implies $y + hg(y) \in C(t+h)$ for all small $h > 0$, hence f satisfies $f(u) \in T_K(t, u)$ for $t \in [0, a)$ and $u \in K(t)$. Consequently, $K(\cdot)$ is positively invariant for $u' + Au = f(u)$ by Lemma 4.2 and Corollary 4.1.
Since $R(\cdot)$ is continuous with $R(\sigma_\infty) + [-\epsilon e, \epsilon e] = C(0)$, there is $\sigma_1 < \sigma_\infty$ such that $R(\sigma_1) \subset C(0)$. Let $t_0 \geq 0$ be such that $\sigma(t_0) > \sigma_1$. Then $u(t_0) \in D(\sigma_1) \subset K(0)$ implies $u(t_0 + a) \in K(a)$. Hence $u(t_0 + a) \in D(\sigma_2)$ for some $\sigma_2 \in (\sigma_\infty, 1)$ with $C(a) = [a + \epsilon e, b - \epsilon e] \subset R(\sigma_2)$ yields the contradiction $\sigma(t_0 + a) > \sigma_\infty$.

Now suppose that $\bigcap_{\tau\in[0,1)} R(\tau) = \{y_\infty\}$ with some $y_\infty \in \mathbb{R}^m$, let $u_\infty(x) = y_\infty$ on Ω and $u(\cdot)$ be the unique mild solution of (5) with $u_0 = u_\infty$ in $D(0)$. Since $u_0 \in D(\tau)$ implies $u(t) \in D(\tau)$ on $\mathbb{R}_+$, we obtain $u(t) \in \bigcap_{\tau\in[0,1)} D(\tau) = \{u_\infty\}$ on $\mathbb{R}_+$, hence $u(t) \equiv u_\infty$ is a stationary solution of (8).

Finally, let u be a mild solution of (8) in $D(0)$ and $\epsilon > 0$. Due to the properties of $R(\cdot)$ there exists $\tau_\epsilon \in [0, 1)$ such that $R(\tau_\epsilon) \subset \overline{B}_\epsilon(y_\infty)$, hence the first part of this proof yields $t_\epsilon \geq 0$ such that $u(t) \in D(\tau_\epsilon)$ on $[t_\epsilon, \infty)$. Therefore $|u_k(t) - u_{\infty,k}|_\infty \leq \epsilon$ for $t \geq t_\epsilon$. □

Let us illustrate the invariance technique by means of a simple example from population dynamics. Consider two competing species sharing the same habitat $\Omega \subset \mathbb{R}^3$, and let u, v denote their population densities. If migration across $\Gamma = \partial\Omega$ is not possible, a model for the time evolution is given by the system

$$\begin{aligned} & u_t = \Delta\varphi(u) + uh_1(u, v) && \text{in } (0, \infty) \times \Omega \\ & v_t = \Delta\psi(v) + vh_2(u, v) && \text{in } (0, \infty) \times \Omega \\ & \frac{\partial\varphi(u)}{\partial\nu} = \frac{\partial\psi(v)}{\partial\nu} = 0 && \text{on } (0, \infty) \times \Gamma \\ & u(0, \cdot) = u_0, \; v(0, \cdot) = v_0 && \text{in } \Omega, \end{aligned} \tag{9}$$

where we assume that Ω, Γ and φ, ψ satisfy the corresponding assumptions of Theorem 6.1. Let us note that nonlinear diffusion of type $\Delta\varphi(u)$ with differentiable φ such that $\varphi'(0) = 0$ and $\varphi'(r) > 0$ otherwise has been proposed in Gurtin/MacCamy [63] for population models in order to take account of an increase in migration due to population pressure (see also §9.3 in Murray [87]). Typical assumptions on h are $h : \mathbb{R}^2_+ \to \mathbb{R}^2$ locally Lipschitz such that $h_1(y_1, \cdot)$, $h_2(\cdot, y_2)$ are decreasing (competition) and

$$\begin{aligned} h_1(y_1, 0) > 0 \text{ in } [0, r_1), \quad h_1(y_1, 0) < 0 \text{ in } (r_1, \infty), \\ h_2(0, y_2) > 0 \text{ in } [0, r_2), \quad h_2(0, y_2) < 0 \text{ in } (r_2, \infty) \end{aligned}$$

with $r_1, r_2 > 0$ (logistic growth of each species in absence of the other one). The assumptions given so far imply that $[0, b] \subset \mathbb{R}^2$ is a contracting rectangle for the ordinary differential equation

$$y' = g(y) \quad \text{with } g(y) = (y_1 h_1(y), y_2 h_2(y)) \tag{10}$$

whenever $b_i > r_i$. Hence the population model (9) admits a global mild solution for every initial value $(u_0, v_0) \in L^\infty(\Omega; \mathbb{R}^2_+)$ by Theorem 6.1, and Theorem 6.2 yields

$$\overline{\lim_{t\to\infty}} |u(t)|_\infty \le r_1, \ \overline{\lim_{t\to\infty}} |v(t)|_\infty \le r_2.$$

Now suppose that h is such that (10) has a unique equilibrium $\overline{y} \in \mathring{\mathbb{R}}^2_+$ (coexistence) which is globally asymptotically stable in $\mathring{\mathbb{R}}^2_+$; let us mention that in the special case

$$h_1(y) = \alpha_1 - \beta_1 y_1 - \gamma_1 y_2, \ \ h_2(y) = \alpha_2 - \beta_2 y_2 - \gamma_2 y_1 \quad \text{with } \alpha_i, \beta_i, \gamma_i > 0,$$

such a globally asymptotically stable equilibrium exists if

$$\Big(\frac{\alpha_1}{\gamma_1} - \frac{\alpha_2}{\beta_2}\Big)\Big(\frac{\alpha_2}{\gamma_2} - \frac{\alpha_1}{\beta_1}\Big) > 0 \quad \text{and} \quad \beta_1\beta_2 > \gamma_1\gamma_2.$$

Under this assumption, the stationary solution $(\overline{u}, \overline{v}) \equiv \overline{y}$ of (9) attracts all mild solutions starting in

$$M = \{(u_0, v_0) \in L^\infty(\Omega; \mathbb{R}^2_+) : \operatorname*{ess\,inf}_\Omega u_0 > 0, \operatorname*{ess\,inf}_\Omega v_0 > 0\}.$$

To see this, let $y(\cdot; y_0)$ denote the unique solution of (10) with initial value $y_0 \in \mathbb{R}^2_+$. Here it is helpful to observe that g is quasimonotone with respect to the cone $C = \mathbb{R}_+ \times \mathbb{R}_-$ due to the monotonicity properties of h_1, h_2, hence $y_0 \le_C z_0$ implies $y(t; y_0) \le_C y(t; z_0)$ on $\mathbb{R}_+$ where $\le_C$ denotes the partial ordering induced by C. Now, given $(u_0, v_0) \in M$, choose $y_0, z_0 \in \mathring{\mathbb{R}}^2_+$ such that $y_0 \le_C \overline{y} \le_C z_0$ and

$$y_{0,1} \le \operatorname*{ess\,inf}_\Omega u_0, \quad y_{0,2} \ge |v_0|_\infty, \quad z_{0,1} \ge |u_0|_\infty, \quad z_{0,2} \le \operatorname*{ess\,inf}_\Omega v_0.$$

Let $y(t) = y(t; y_0)$ and $z(t) = y(t; z_0)$. Then $y(t) \le_C \overline{y} \le_C z(t)$ on $\mathbb{R}_+$, $y(t) \to \overline{y}$ as well as $z(t) \to \overline{y}$ as $t \to \infty$, and $R(t) = \{\eta \in \mathbb{R}^2_+ : y(t) \le_C \eta \le_C z(t)\}$ defines a positively invariant tube for (10). Evidently the $R(t)$ are rectangles. Here, instead of showing that the $R(t)$ are also contracting, it is easier to apply directly Corollary 4.2 to conclude that

$(u(t), v(t)) \in K(t)$ for $t \geq 0$ with $K(t) = \{(u,v) \in L^1(\Omega; \mathbb{R}^2) : (u(x), v(x) \in R(t)$ a.e. on $\Omega\}$. Hence $(u(t), v(t)) \to (\overline{u}, \overline{v})$ in $L^\infty(\Omega)^2$.

This example shows how a direct comparison between solutions of reaction-diffusion systems like (4) or (8) and of the associated ordinary differential equation $y' = g(t,y)$ is possible if g is quasimonotone with respect to a cone that induces rectangular order intervals. This well-known fact may for instance be used to formulate sufficient conditions for existence of global solutions. A typical result in this direction is

Proposition 6.1 *Let $\Omega \subset \mathbb{R}^n$ be open bounded with C^2-boundary Γ, $\varphi_k : \mathbb{R} \to \mathbb{R}$ be continuous, strictly increasing with $\varphi_k(0) = 0$ and $g : \mathbb{R}_+ \times \mathbb{R}_+^m \to \mathbb{R}^m$ be continuous with $g_k(t,y) \geq 0$ if $y_k = 0$. Define $\overline{g} : \mathbb{R}_+ \times \mathbb{R}_+^m \to \mathbb{R}^m$ by*

$$\overline{g}_k(t,y) = \max\{g_k(t,z) : z_k = y_k, 0 \leq z_i \leq y_i \text{ for } i \neq k\}.$$

Given $u_0 \in L^\infty(\Omega; \mathbb{R}_+^m)$, let $y_0 = (|u_{0,1}|_\infty, \dots, |u_{0,m}|_\infty)$ and $\overline{y}(\cdot)$ be a (local) solution of

$$y' = \overline{g}(t,y) \quad \text{on } \mathbb{R}_+, \quad y(0) = y_0. \tag{11}$$

Suppose that $\overline{y}$ has a maximal extension onto $[0,T)$ for some $T \in (0,\infty]$. Then the reaction-diffusion systems (4) and (8) with initial value u_0 admit mild solutions on $[0,T)$.

Proof. Evidently $\overline{g}$ is continuous with $\overline{g}_k(t,y) \geq 0$ if $y_k = 0$, hence (11) has a local solution $\overline{y}(\cdot)$ which admits a maximal extension onto $[0,T)$ for some $T \in (0,\infty]$. Let $C(t) = [0, \overline{y}(t)] \subset \mathbb{R}_+^m$,

$$K(t) = \{u \in L^1(\Omega)^m : u(x) \in C(t) \text{ a.e. on } \Omega\} \quad \text{on } [0,T)$$

and notice that $\mathrm{gr}\,(K)$ is closed from the left. Define $f : \mathrm{gr}\,(K) \to L^1(\Omega)^m$ by $f(t,u)(x) = g(t, u(x))$ for $x \in \Omega$. Inspection of the proof of Theorem 6.1 shows that the abstract initial value problem (6) corresponding to (4) or (8) admits a mild solution on every interval $[0,a] \subset [0,T)$, given that $f(t,u) \in T_K(t,u)$ on $\mathrm{gr}\,(K)$. Furthermore, the latter holds if

$$\lim_{h \to 0+} h^{-1} \rho(y + hg(t,y), C(t+h)) = 0 \quad \text{for all } t \in [0,T), y \in C(t).$$

Let $t \in [0,T)$ and $y \in C(t)$. Then $y_k = 0$ implies $y_k + hg_k(t,y) \geq 0$, while $y_k = \overline{y}_k(t)$ yields $g_k(t,y) \leq \overline{g}_k(t, \overline{y}(t)) = \overline{y}_k'(t)$ by definition of $\overline{g}$, hence $y_k + hg_k(t,y) \leq \overline{y}_k(t+h) + o(h)$. Consequently, $\lim_{h \to 0+} h^{-1} \rho(y_k + hg_k(t,y), [0, \overline{y}_k(t)]) = 0$ which ends the proof. $\square$

Observe that $\overline{g} = g$ holds in the situation of Proposition 6.1 iff g is quasimonotone with respect to $\mathbb{R}_+^m$, i.e. iff all components $g_k(t,\cdot)$ are increasing in y_i for every $i \neq k$.

There are other applications which lead to reaction-diffusion systems with discontinuous reaction terms. Such situations occur for instance if certain limiting cases are considered, where the discontinuous "law" g appears as an approximation of a more complicated, maybe continuous or locally Lipschitz model. To explain this in more detail, consider the specific case of a single irreversible exothermic reaction, taking place inside a bounded region $\Omega \subset \mathbb{R}^3$. Under several simplifying assumptions this leads to the following mathematical model; see Chapter 2.5 in Aris [4].

$$\begin{aligned} u_t &= \Delta u - \mu^2 h(u) \exp\left(\gamma \frac{v-1}{v}\right) && \text{in } (0,T) \times \Omega \\ \lambda v_t &= \Delta v + \sigma \mu^2 h(u) \exp\left(\gamma \frac{v-1}{v}\right) && \text{in } (0,T) \times \Omega \\ u &= v = 1 && \text{on } (0,T) \times \Gamma \\ u(0,\cdot) &= u_0,\ v(0,\cdot) = v_0 && \text{in } \Omega. \end{aligned}$$

Here u and v denote the concentration of the reactand, respectively the temperature in dimensionless form, and the boundary condition refers to the case when both are constant in the surrounding bulk phase. Concerning the constants appearing in this model, let us only note that μ is the Thiele number, $\sigma > 0$ the Prater temperature, $\gamma > 0$ the Arrhenius number and $\lambda^{-1} > 0$ is the Lewis number. The reaction rate h is of course only meaningful for nonnegative concentrations and a typical choice is $h(r) = r^\alpha$ with $\alpha > 0$ in case of "reactions of order α"; see, e.g. Chapter 2 in Espenson [50]. In the limiting case $\alpha = 0$ of zero-order reactions this leads to the Heaviside function, i.e. $h(0) = 0$ and $h(r) = 1$ for $r > 0$.

In practice, chemical reactions are often performed inside porous catalytic pellets, hence nonlinear diffusion inside Ω has to be taken into account for a more realistic model. This leads to a reaction-diffusion system of type (4) but with a discontinuous reaction term $g : \mathbb{R}^m_+ \to \mathbb{R}^m$, and the example above shows that, in general, we cannot expect g to satisfy any condition of dissipative type.

Other examples with discontinuous nonlinearities arise, for instance, in porous medium combustion in the limiting case of large activation energy (see Norbury/Stuart [88], [89] and Stuart [104] as well as Remark 6.3), in climate modelling where the discontinuity is due to a jump of the planetary coalbedo as a function of the temperature (see Diaz [44]), and in the instantaneous limiting case of irreversible concurring reactions (see §7 below).

It has been mentioned before that ordinary differential equations, say $y' = g(y)$ with $g : \mathbb{R}^m \to \mathbb{R}^m$, need not admit (local) solutions in case of discontinuous right-hand sides. On the other hand, if g is obtained as the approximation of a continuous function h and if the graph of h is, in a certain sense, sufficiently close to the graph of g, then a given solution of $y' = h(y)$ with initial value $y(0) = y_0$ will be close to some solution of $y' \in G(y)$, $y(0) = y_0$, where the multivalued "regularization" G is given by

$$G(y) = \bigcap_{\delta > 0} \overline{\text{conv}}\, g(B_\delta(y)).$$

The following example from Applied Mechanics illustrates this passage to a modified right-hand side. Suppose that an oscillating mass is subjected to dry friction forces due to its contact to a wall, say. This leads for instance to differential equations of the type

$$y'' + h(y') + \mu(y)\,\mathrm{sgn}\, y' + f(y) = \varphi(t)$$

where h, f and φ correspond to viscous damping, restoring and external forces, respectively. Furthermore $-\mu(y)\,\mathrm{sgn}\, y'$ is a "law" to describe dry friction forces, called Coulomb's law in case of constant μ. Here an interesting phenomenon due to the presence of dry friction is the occurence of so-called stick-slip-motions, i.e. solutions may have deadzones ("stick"), say $y(t) \equiv c$ on some interval $[\sigma, \tau]$. This means $h(0) + \mu(c)\,\mathrm{sgn}\,(0) + f(c) = \varphi(t)$ on $[\sigma, \tau]$, hence one cannot defined $\mathrm{sgn}\,(0)$ as a fixed value but has to allow the whole interval $[-1, 1]$. Hence the problem has to be modeled by the differential inclusion

$$y'' + h(y') + \mu(y)\,\mathrm{Sgn}\, y' + f(y) \ni \varphi(t)$$

instead, which corresponds to the passage from a discontinuous g to its usc regularization G mentioned above. For more details and mathematical results concerning differential inclusions of this particular type see Bothe [24], Deimling/Hetzer/Shen [43] and the references given in these papers; concerning the whole subject of discontinuous differential equations see §A.1 in Deimling [42] and Filippov [54].

It is therefore reasonable to replace g in (4) by G, where $B_\delta(y)$ has to be replaced by $B_\delta(y) \cap \mathbb{R}^m_+$ in the definition of G above in case g is only defined on $\mathbb{R}^m_+$. Under the mild assumption that g is locally bounded it follows that $G : \mathbb{R}^m_+ \to 2^{\mathbb{R}^m} \setminus \emptyset$ is usc with compact convex values and $G(y) = \{g(y)\}$ if g is continuous at y; remember §2.1. Consequently, we are led to consider

$$u_t \in \Delta\Phi(u) + G(u) \ \text{ in } (0,\infty) \times \Omega, \ \ \Phi(u) = 0 \ \text{ on } (0,\infty) \times \Gamma, \ \ u(0,\cdot) = u_0 \ \text{ in } \Omega, \tag{12}$$

where $u = (u_1, \ldots, u_m)$ and $\Delta\Phi(u) = (\Delta\varphi_1(u_1), \ldots, \Delta\varphi_m(u_m))$. Let us note in passing that a componentwise notation would be misleading here, since $z_k \in G_k(y)$ for $k = 1, \ldots, m$ is not the same as $z \in G(y)$ unless $G(y)$ is a rectangle.
Problem (12) will be considered in its abstract formulation as the nonlinear evolution problem

$$u' \in -Au + F(u) \ \text{ on } \mathbb{R}_+, \ \ u(0) = u_0 \tag{13}$$

with multivalued F of usc type. Therefore, let us first clarify how much regularity can be expected for F, given that G is usc.

Proposition 6.2 *Let $G : \mathbb{R}^m_+ \to 2^{\mathbb{R}^m} \setminus \emptyset$ be usc with compact convex values, $\Omega \subset \mathbb{R}^n$ be measurable and bounded and let $p \in [1, \infty)$. Then $F : L^p(\Omega; \mathbb{R}^m_+) \to 2^{L^p(\Omega)^m}$, defined by*

$$F(u) = \{v \in L^p(\Omega)^m : v(x) \in G(u(x)) \ \text{a.e. on } \Omega\},$$

has nonempty, weakly compact and convex values for all $u \in L^\infty(\Omega; \mathbb{R}^m_+)$. Moreover, F is ϵ-δ-usc on every $|\cdot|_\infty$-bounded subset of $L^p(\Omega; \mathbb{R}^m_+)$.

Proof. Let $u \in L^\infty(\Omega;\mathbb{R}^m_+)$. Then $G(u(\cdot)) : \Omega \to 2^{\mathbb{R}^m} \setminus \emptyset$ is measurable, since $G(u(\cdot))^{-1}(A) = \{x \in \Omega : u(x) \in G^{-1}(A)\}$ is a measurable subset of Ω for every closed $A \subset \mathbb{R}^m$. Having also closed values, $G(u(\cdot))$ admits a measurable selection v by Lemma 2.2, and $v \in L^\infty(\Omega)^m \subset L^p(\Omega)^m$ since G is bounded on bounded sets. Consequently $F(u) \neq \emptyset$, and the other properties of the values $F(u)$ are obvious.

Let $(u_k) \subset L^p(\Omega;\mathbb{R}^m_+)$ with $M := \sup\limits_{k\geq 1} |u_k|_\infty < \infty$ such that $u_k \to u$ in $L^p(\Omega)^m$. Given $\epsilon > 0$, we then have to show that

$$F(u_k) \subset F(u) + B_\epsilon(0) \quad \text{ for all } k \geq k_\epsilon.$$

We may assume $u_k \to u$ a.e. on Ω. By the theorems of Lusin (together with its multivalued version) and Egorov, given $\sigma > 0$, there exists a closed $\Omega_\sigma \subset \Omega$ with $\lambda_n(\Omega \setminus \Omega_\sigma) \leq \sigma$ such that $G(u(\cdot))|_{\Omega_\sigma}$, $u|_{\Omega_\sigma}$ as well as all $u_k|_{\Omega_\sigma}$ are continuous and $\sup\limits_{x\in\Omega_\sigma} |u_k(x) - u(x)| \to 0$ as $k \to \infty$. We claim that, given also $\eta > 0$, there is $k_0 = k_0(\eta,\sigma) \geq 1$ such that

$$G(u_k(x)) \subset G(u(x)) + B_\eta(0) \quad \text{on } \Omega_\sigma \qquad \text{for all } k \geq k_0. \tag{14}$$

Suppose not. Then there is $(x_j) \subset \Omega_\sigma$ such that $G(u_{k_j}(x_j)) \not\subset G(u(x_j)) + B_\eta(0)$ for all $j \geq 1$, where $k_j \nearrow \infty$. Without loss of generality we have $x_j \to x_0 \in \Omega_\sigma$, hence $u_{k_j}(x_j) \to u(x_0)$ implies $G(u_{k_j}(x_j)) \subset G(u(x_0)) + B_{\eta/2}(0)$ for all sufficiently large j. Therefore, by continuity of $G(u(\cdot))|_{\Omega_\sigma}$, we get the contradiction $G(u_{k_j}(x_j)) \subset G(u(x_j)) + B_\eta(0)$ for all large j.
Since $G(\overline{B}_M(0) \cap \mathbb{R}^m_+) \subset \overline{B}_R(0)$ for some $R > 0$ and

$$\rho(v_k, F(u))^p = \int_\Omega \rho(v_k(x), G(u(x)))^p \, dx \quad \text{for } v_k \in F(u_k),$$

exploitation of (14) yields

$$\rho(v_k, F(u)) \leq \Big(\eta^p \lambda_n(\Omega_\sigma) + (2R)^p \lambda_n(\Omega \setminus \Omega_\sigma)\Big)^{1/p} \leq \Big(\eta^p \lambda_n(\Omega) + (2R)^p \sigma\Big)^{1/p}$$

for any $v_k \in F(u_k)$ with $k \geq k_0$. Hence the second assertion follows by choosing $\sigma, \eta > 0$ such that $\eta^p \lambda_n(\Omega) + (2R)^p \sigma < \epsilon^p$. □

While it is rather obvious that F is weakly usc on $|\cdot|_\infty$-bounded subsets in the situation of Proposition 6.2, such F need not be usc. This is shown by the following counter-example.

Example 6.1 Let $\Omega = (-1,1)$, $G : \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ with

$$G(0) = [-1,1] \text{ and } G(r) = \{r + \operatorname{sgn}(r)\} \text{ for } r \neq 0,$$

and define $F : L^1(\Omega) \to 2^{L^1(\Omega)} \setminus \emptyset$ by $F(u) = \{v \in L^1(\Omega) : v(x) \in G(u(x)) \text{ a.e. on } \Omega\}$. Let $v_n = (1 + 1/n)r_n$, where $r_n(x) = \operatorname{sgn}(\sin(2^n x))$ are the Rademacher functions. Then $A := \{v_n : n \geq 1\}$ is a closed subset of $L^1(\Omega)$, since $v_n \rightharpoonup 0$ and no subsequence (v_{n_k}) satisfies $v_{n_k}(x) \to 0$ a.e. on Ω. Evidently $F^{-1}(A) = \{\frac{1}{n} r_n : n \geq 1\}$ is not closed, hence F is not usc on $|\cdot|_\infty$-bounded subsets of $L^1(\Omega)$. ◇

By means of Proposition 6.2 we are able to prove

Theorem 6.3 *Let $\Omega \subset \mathbb{R}^n$ be open bounded with C^2-boundary Γ, $\varphi_k : \mathbb{R} \to \mathbb{R}$ be continuous, strictly increasing with $\varphi_k(0) = 0$ and $G : \mathbb{R}^m_+ \to 2^{\mathbb{R}^m} \setminus \emptyset$ be usc with compact convex values. Let $T \in (0, \infty]$, $\overline{y} : [0, T) \to \mathbb{R}^m_+$ be continuously differentiable and $C(t) = [0, \overline{y}(t)]$ on $[0, T)$. Suppose that $G(y) \cap T_C(t, y) \neq \emptyset$ on $gr\,(C)$. Then, given $u_0 \in L^\infty(\Omega; \mathbb{R}^m_+)$ with $|u_{0,k}|_\infty \leq \overline{y}_k(0)$ for $k = 1, \ldots, m$, problem (12) has a mild solution u on $[0, T)$ such that $0 \leq u_k(t, x) \leq \overline{y}_k(t)$ a.e. on Ω for all $t \in [0, T)$ and $k = 1, \ldots, m$.*
In particular, if $G(y) \cap T_{\mathbb{R}^m_+}(y) \neq \emptyset$ on $\mathbb{R}^m_+$ then (12) has a local nonnegative mild solution for every $u_0 \in L^\infty(\Omega; \mathbb{R}^m_+)$.

Proof. 1. We consider (12) in its abstract formulation as the initial value problem (13) in $X = L^1(\Omega)^m$ where Au corresponds to $(-\Delta\varphi_1(u_1), \ldots, -\Delta\varphi_m(u_m))$ with Dirichlet boundary conditions; cf. the proof of Theorem 6.1. Let

$$K(t) = \{u \in X : u(x) \in C(t) \text{ a.e. on } \Omega\} \text{ on } [0, T),$$

and define $F : \mathrm{gr}\,(K) \to 2^X \setminus \emptyset$ by means of

$$F(u) = \{v \in X : v(x) \in G(u(x)) \text{ a.e. on } \Omega\}.$$

We are going to apply Theorem 4.7, and it suffices to show that (13) has a mild solution on $J = [0, a]$ with arbitrary $a \in (0, T)$. The following assumptions of Theorem 4.7 are obviously satisfied: $\mathrm{gr}\,(K)$ is closed from the left, F is bounded on $\mathrm{gr}\,(K_{|J})$ and maps bounded sets into weakly relatively compact sets; in fact F maps $\mathrm{gr}\,(K_{|J})$ into an L^∞-bounded subset of X. Moreover F is ϵ-δ-usc with weakly compact convex values by Proposition 6.2, and condition (36) in front of Theorem 4.7 is satisfied (for $u_0 \in L^\infty(\Omega)^m$) due to Lemma 6.2(b). Hence (13) has a mild solution u on J with $u(\cdot) \in K(\cdot)$ by Theorem 4.7 and Remark 4.3 if $F(u) \cap T_K(t, u) \neq \emptyset$ on $\mathrm{gr}\,(K)$ and $T_K(\cdot, \cdot)$ is lsc with closed convex values.
To check these remaining conditions, notice first that

$$T_C(t, y) = \{z \in \mathbb{R}^m : z_k \geq 0 \text{ if } y_k = 0,\ z_k \leq \overline{y}'_k(t) \text{ if } y_k = \overline{y}_k(t)\} \text{ on } \mathrm{gr}\,(C),$$

and

$$\lim_{h \to 0+} \frac{1}{h} \rho(y + hz, C(t + h)) = 0 \quad \text{if } z \in T_C(t, y) \text{ with } (t, y) \in \mathrm{gr}\,(C) \tag{15}$$

since $\overline{y}'$ is continuous. The latter also implies that $T_C(\cdot, \cdot)$ is lsc, and $T_C(t, y)$ is obviously closed convex. Fix $t \in [0, T)$, $u \in K(t)$ and observe that $G(u(\cdot)) \cap T_C(t, u(\cdot))$ is measurable with nonempty values. The latter holds since both $G(u(\cdot))$ and $T_C(t, u(\cdot))$ are measurable; for instance

$$T_C(t, u(\cdot))^{-1}(V) = \{x \in \Omega : u(x) \in T_C(t, \cdot)^{-1}(V)\},$$

and $T_C(t,\cdot)^{-1}(V)$ is open for all open sets V since $T_C(t,\cdot)$ is lsc. Hence application of Lemma 2.2 yields a measurable selection v of $G(u(\cdot)) \cap T_C(t,u(\cdot))$. Due to

$$\frac{1}{h}\rho(u+hv, K(t+h)) = \int_\Omega \frac{1}{h}\rho(u(x)+hv(x), C(t+h))dx,$$

we obtain $v \in T_K(t,u)$ by means of (15) and the dominated convergence theorem. Hence $F(u) \cap T_K(t,u) \neq \emptyset$ since $v \in F(u)$. In fact the same argument yields

$$T_K(t,u) = \{v \in X : v(x) \in T_C(t,u(x)) \text{ a.e. on } \Omega\} \text{ on gr}\,(K),$$

which immediately shows that the $T_K(t,u)$ are closed convex. It remains to prove lower semicontinuity of $T_K(\cdot,\cdot)$ which follows if $\rho(v, T_K(\cdot,\cdot))$ is usc for all $v \in X$. For this purpose let $(t_k) \subset [0,T)$ with $t_k \to t \in [0,T)$, $u^k \in K(t_k)$ with $u^k \to u$ and $v \in X$, where we may assume $u^k(x) \to u(x)$ a.e. on Ω. Then

$$\overline{\lim_{k\to\infty}}\, \rho(v, T_K(t_k,u^k)) = \overline{\lim_{k\to\infty}} \int_\Omega \rho(v(x), T_C(t_k,u^k(x)))dx$$
$$\le \int_\Omega \overline{\lim_{k\to\infty}}\, \rho(v(x), T_C(t_k,u^k(x)))dx \le \int_\Omega \rho(v(x), T_C(t,u(x)))dx = \rho(v, T_K(t,u));$$

for the first inequality notice that $\rho(v(x), T_C(t_k,u^k(x))) \le |v(x)| + M$ a.e. on Ω where $M = \sup_k |\overline{y}'(t_k)|$, since $T_C(t,y)$ contains an element z with $|z| \le |\overline{y}'(t)|$ for every $(t,y) \in \text{gr}\,(C)$. Hence $T_K(\cdot,\cdot)$ is lsc, which ends the proof of the first assertion.

2. Given $u_0 \in L^\infty(\Omega;\mathbb{R}^m_+)$, let $y_0 = (|u_{0,1}|_\infty, \ldots, |u_{0,m}|_\infty)$. Since the usc G is bounded on bounded sets, there is $M > 0$ such that $||G(y)|| \le M$ on $[0, y_0 + e]$ with $e = (1,\ldots,1) \in \mathbb{R}^m$. Let $\overline{y}(t) = y_0 + tMe$ and $C(t) = [0, \overline{y}(t)]$ on $[0,a]$ with $a = 1/M$. Given $t \in [0,a)$ and $y \in C(t)$ there is $z \in G(y) \cap T_{\mathbb{R}^m_+}(y)$, and $|z| \le M$. This implies $y + hz \in C(t+h)$ for all small $h > 0$, hence $z \in T_C(t,y)$. Therefore (12) has a nonnegative mild solution on $[0,a)$ by step 1 of this proof. $\square$

6.3 A problem from heterogeneous catalysis

The subsequent model problem plays a fundamental role in chemical engineering within the context of heterogeneous catalytic reactors. The underlying situation is that a chemical reaction, say $A + B \to P$ in the simplest case, is to be performed which only takes place (with economic speed) if being catalysed. In the majority of industrial applications the catalytic substance forms a separate phase and the catalyst frequently comes in the form of porous pellets. In this case the process is said to be heterogeneously catalysed, since the overall system consists of at least two different phases. The actual number of different phases depends of course on the specific type of process under consideration. In the sequel we consider a model that refers, for instance, to so-called stirred slurry reactors, the type probably encountered most frequently in industrial practice. In this reactor design solid particles of small size (the porous catalytic pellets) are suspended in a liquid bulk phase either mechanically, or by means of gas bubbling. As a result, the reactor is almost perfectly mixed with respect to the liquid and can be operated isothermally. The appearance of an additional gas phase is typical, since the reactants are often fed into the reactor in different phases. Still it suffices to consider a two-phase liquid/solid system, given that transition of the species from gas to liquid is comparatively fast, a reasonable assumption in many concrete applications.

In order that reaction takes place, A and B have to diffuse into the interior of the catalyst to reach the active sites at the inner surface of the porous pellets. For a realistic model of the overall process one has to take into account macroscopic convection, interfacial mass transfer as well as diffusion, adsorption and reaction within the pellets. In mathematical terms the following system of coupled nonlinear reaction-diffusion equations with dynamical boundary conditions results.

$$
\begin{aligned}
&\frac{\partial c_A}{\partial t} = \Delta\varphi_A(c_A) - r(c_A, c_B) \ \text{ in } \Omega, \qquad \frac{\partial \varphi_A(c_A)}{\partial \nu} = \gamma_A(c_A^b - h_A(c_A)) \ \text{ on } \Gamma \\
&\frac{\partial c_B}{\partial t} = \Delta\varphi_B(c_B) - r(c_A, c_B) \ \text{ in } \Omega, \qquad \frac{\partial \varphi_B(c_B)}{\partial \nu} = \gamma_B(c_B^b - h_B(c_B)) \ \text{ on } \Gamma \\
&\frac{dc_A^b}{dt} = \rho(c_A^f(t) - c_A^b) - \int_\Gamma \gamma_A(c_A^b - h_A(c_A))d\sigma \\
&\frac{dc_B^b}{dt} = \rho(c_B^f(t) - c_B^b) - \int_\Gamma \gamma_B(c_B^b - h_B(c_B))d\sigma
\end{aligned}
\tag{16}
$$

Here we assume that all pellets are of the same shape given by a certain set $\Omega \subset \mathbb{R}^3$ with C^2-boundary Γ, and ν denotes the outer normal. To take care of the pellets' high porosity, diffusion within the pellets is modelled by $\Delta\varphi_A(c_A)$ and $\Delta\varphi_B(c_B)$, respectively, with continuous strictly increasing functions φ_A and φ_B. Evolution of the bulk concentrations is described by the latter set of equations, where the positive constant ρ denotes the ratio $\dot{V}_L^f/V_L$ between the liquid flow rate $\dot{V}_L^f$ and the liquid volume V_L, and the feeds c_A^f, c_B^f are time-dependent nonnegative functions. The integral terms reflect mass transport into the pellets; actually the integrals would appear with the factor N/V_L in front, where N is the total number of pellets, but this can be omitted after rescaling the bulk concentrations. Interfacial mass transport is typically modelled by so-called film-diffusion, where it is assumed that a stagnant boundary layer is present around the pellets, separating the pellets from the region of turbulent liquid. At the outer surface of this film the concentration of a specific species equals the corresponding bulk concentration, while at the inner surface continuity of mass-flow (i.e. $\partial_\nu\varphi_j(c_j) = D_j\partial_\nu c_j$) as well as thermodynamic equilibrium is assumed. The latter typically leads to a jump of the concentrations at the surface of the pellets, which enters the model in terms of increasing functions h_A, h_B. The particular boundary condition used in (16) follows from additional simplifying assumptions which are reasonable if the boundary layer is sufficiently thin. In this case we may assume stationary diffusion within the film, hence if c_j denotes the concentration of species j in the film then $\partial_\nu c_j(x)$ is approximately given by $\gamma(c_j^b - c_j(x))$, where $1/\gamma$ is the thickness of the film. This approximation becomes exact in the radial symmetric case, where γ is a different constant which also depends on the diameter of the pellets. Without these simplifications an additional set of equations for the film-diffusion appears and leads to a slightly more complicated model, but which still can be handled by the same methods.

Finally some explanations concerning the reaction term are in order. Intrinsic to the model is the basic assumption that the catalytic sites are distributed uniformly over the whole pellet; in fact the inner surface is identified with Ω. This is reasonable since any efficient catalyst necessarily has to be endowed with an extremely large active surface. There are two possible mechanisms for a catalysed reaction $A+B \to P$. In the first case it suffices if a molecule of one of the involved species (say A) is adsorbed at a catalytic site, to enable the chemical reaction with a molecule B. This leads to a chain of two "reactions", namely $A \rightleftharpoons A^*$ and $A^* + B \to P$ where A^* denotes the adsorbed species. Here we assume that adsorption is fast compared to reaction and diffusion, which is often valid in practice. In this situation, adsorption of A can be modelled as a quasi-stationary process, hence the concentration c_{A^*} of the adsorbed species is given as $c_{A^*} = q(c_A)$, where the function q is continuous, increasing and bounded with $q(0) = 0$. The boundedness of q refers to a saturation effect inside the pellet due to the limited number of available catalytic sites. Consequently, if $A^* + B \to P$ takes place with rate $r_0(c_{A^*}, c_B)$, the overall reaction has rate $r(c_A, c_B) = r_0(q(c_A), c_B)$. A second mechanism appears if molecules of both species must be adsorbed to allow for the chemical reaction. In this case the adsorped concentrations may depend on both c_A and c_B, since the adsorption processes may interfere. This leads to a reaction rate of type $r(c_A, c_B) = r_0(q_A(c_A, c_B), q_B(c_A, c_B))$ with continuous functions q_A, q_B satisfying $q_A(0, c_B) = q_B(c_A, 0) = 0$. However, in each case we end up with a continuous reaction rate $r(\cdot, \cdot)$ which is defined for nonnegative concentrations, satisfying $r(c_A, c_B) \geq 0$ and $r(c_A, c_B) = 0$ if $c_A c_B = 0$. Notice that any realistic reaction rate $r_0(\cdot, \cdot)$, like $r_0(c_A, c_B) = k c_A^\alpha c_B^\beta$ with $\alpha, \beta > 0$ to mention a typical example, has this latter property which is then inherited by $r(\cdot, \cdot)$. Therefore, concerning the analysis of the mathematical model, no distinction is made. Finally, the minus sign in front of $r(c_A, c_B)$ in (16) refers to the fact that both A and B are educts.

More information on chemical reaction engineering is given e.g. in [76] and [58], while further details on heterogeneous catalysis including adsorption can be found for instance in [5] and [115].

In several concrete cases the following additional features occur:

(i) The feeds may vary periodically with time either due to external fluctuations or because the process is operated in a periodic manner. While the first case happens for instance in biochemical systems like waste water treatment plants (in which case agglomerates of microorganisms play the role of the catalytic pellets), the second situation is quite common since periodic control (also called "forced cycling" within this context) is often used to increase the performance of such processes with respect to conversion or selectivity. An explanation for this phenomenon is given in [11] for an isothermal second-order reaction $2A \to P$ performed in a continuous-flow stirred tank reactor (CSTR). The basic idea is as follows: In case of a constant feed the concentration of the product tends to the unique stationary solution, which is a convex function of the feed concentration. Hence the average performance over a cycle is greater than the performance for the correspondingly averaged feed.

Let us only mention two more references from the extensive literature related to periodic operation of chemical reactors. An interesting, although quite general discussion about the advantages of forced cycling from a chemical engineering viewpoint is given in [83], while a summary of some theoretical results for problems involving (linear) diffusion and reaction can be found in Chapter 8.5 in [4].

(ii) If variations of the temperature are present the reaction rates will also depend on time, and if these fluctuations are due to seasonal changes in the external environment this leads to T-periodic reaction rates.

Therefore, besides questions of existence and uniqueness, the problem of existence of a T-periodic solution appears naturally. In the present section we study this question for the following reaction-diffusion system, which is of the same type as (16) but allows for m involved species as well as time-dependent feeds and reaction rates.

$$\begin{aligned}
\frac{\partial v_k}{\partial t} &= \Delta\varphi_k(v_k) + r_k(t,x,v_1,\ldots,v_m) && \text{for } t>0,\ x\in\Omega \\
\frac{\partial \varphi_k(v_k)}{\partial \nu} &= \gamma_k(c_k - h_k(v_k)) && \text{for } t>0,\ x\in\Gamma \\
\frac{dc_k}{dt} &= -\int_\Gamma \gamma_k(c_k - h_k(v_k))d\sigma + R_k(t,c_1,\ldots,c_m) && \text{for } t>0
\end{aligned} \tag{17}$$

$k = 1,\ldots,m.$

In the sequel, we consider (17) under the following assumptions: $\gamma_k > 0$, $\varphi_k : \mathbb{R} \to \mathbb{R}$ continuous and strictly increasing with $\varphi_k(0) = 0$, $h_k : \mathbb{R} \to \mathbb{R}$ continuous, strictly increasing and onto with $h_k(0) = 0$, $r_k : \mathbb{R} \times \Omega \times \mathbb{R}^m_+ \to \mathbb{R}$ and $R_k : \mathbb{R} \times \mathbb{R}^m_+ \to \mathbb{R}$ T-periodic and of Carathéodory type. We consider problem (17) in an abstract formulation where it is appropriate to choose an L^1-setting in order to exploit conservation of mass. For this purpose define the operator A in $X = L^1(\Omega) \times \mathbb{R}$ (with norm $|(v,c)| = |v|_1 + |c|$) by

$$\begin{aligned}
&A\begin{pmatrix} v \\ c \end{pmatrix} = \begin{pmatrix} -\Delta\varphi(v) \\ \int_\Gamma \gamma(c - h(v))d\sigma \end{pmatrix} \quad \text{for } (v,c) \in D(A), \text{ where} \\
&D(A) = \{(v,c) \in X : \varphi(v) \in W^{1,1}(\Omega), \Delta\varphi(v) \in L^1(\Omega), \frac{\partial\varphi(v)}{\partial\nu} = \gamma(c - h(v)) \text{ on } \Gamma\}.
\end{aligned} \tag{18}$$

With this definition the reaction-diffusion system (17) has the abstract formulation

$$u_k' + A_k u_k = f_k(t,u) \text{ on } \mathbb{R}_+ \qquad (k = 1,\dots,m). \tag{19}$$

Here $u_k = (v_k, c_k)$, A_k is the operator given by (18) with φ_k, h_k and γ_k instead of φ, h and γ, respectively, and $f_k(t,u) = (r_k(t,\cdot,v(\cdot)), R_k(t,c))$.
In analogy to definition (2), the exact formulation of (18) reads as

$$(\begin{pmatrix} v \\ c \end{pmatrix}, \begin{pmatrix} g \\ r \end{pmatrix}) \in \text{gr}(A) \text{ iff } w := \gamma(c - h(v)) \in L^1(\Gamma) \text{ with } \int_\Gamma w d\sigma = r$$

$$\text{and } u = \varphi(v) \text{ is the weak solution of } \; -\Delta u = g \text{ in } \Omega, \; \frac{\partial u}{\partial \nu} = w \text{ on } \Gamma, \tag{20}$$

where $v_{|\Gamma}$ is understood as $\varphi^{-1}(\varphi(v)_{|\Gamma})$. To establish an appropriate compactness property of the semigroup generated by $-A$, we shall use the following characterization of Lyapunov functions, which is a special case of Theorem 19.3 in [17].

Lemma 6.3 *Let A be m-accretive in a real Banach space X and $M, N : \mathbb{R}_+ \times \overline{D(A)} \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous such that*

$$N(t+\lambda, J_\lambda u) + \lambda M(t, J_\lambda u)) \le N(t,u) \text{ for all } t \ge 0,\ u \in \overline{D(A)} \text{ and small } \lambda > 0.$$

Then

$$N(s+t, S(t)u) + \int_0^t M(s+\tau, S(\tau)u)d\tau \le N(s,u) \text{ for all } s,t \ge 0 \text{ and } u \in \overline{D(A)},$$

where $S(t)$ denotes the semigroup generated by $-A$.

The next results provides several properties of the operator A given by (18).

Lemma 6.4 *Let $\Omega \subset \mathbb{R}^n$ be open bounded with C^2-boundary Γ and $X = L^1(\Omega) \times \mathbb{R}$ equipped with the partial ordering $(v,c) \le (\overline{v}, \overline{c})$ if $v \le \overline{v}$ a.e. on Ω and $c \le \overline{c}$. Let A be the operator in X defined by (18) with $\gamma > 0$, $\varphi : \mathbb{R} \to \mathbb{R}$ continuous, strictly increasing with $\varphi(0) = 0$ and $h : \mathbb{R} \to \mathbb{R}$ continuous, strictly increasing and onto with $h(0) = 0$. Then A is m-accretive and T-accretive with $\overline{D(A)} = X$, and the resolvent satisfies*

$$\begin{pmatrix} \min\{-|v^-|_\infty, h^{-1}(c)\} \\ \min\{h(-|v^-|_\infty), c\} \end{pmatrix} \le J_\lambda \begin{pmatrix} v \\ c \end{pmatrix} \le \begin{pmatrix} \max\{|v^+|_\infty, h^{-1}(c)\} \\ \max\{h(|v^+|_\infty), c\} \end{pmatrix} \} \text{ for all } \lambda > 0. \tag{21}$$

Moreover, $S(t)B$ is relatively compact for $t > 0$ whenever $B \subset X^+$ is weakly relatively compact, where $X^+ = \{(v,c) \in X : v, c \ge 0\}$ and $S(\cdot)$ denotes the semigroup generated by $-A$.

Proof. To keep the proof readable we use formal computations (in particular partial integration) at several places. To make those arguments precise, one may use the following fact: Let $g \in L^1(\Omega)$, $w \in L^1(\Gamma)$, $1 \le q < \frac{n}{n-1}$ and u be a weak solution of (20). Then there is

$(u_n) \subset C^2(\overline{\Omega})$ such that $u_n \to u$ in $W^{1,q}(\Omega)$, $-\Delta u_n \to g$ in $L^1(\Omega)$ and $\dfrac{\partial u_n}{\partial \nu} \to w$ in $L^1(\Gamma)$; this is a consequence of Theorem 22 and Lemma 23 in Brezis/Strauss [32].
Evidently $\{(v,c) \in X : \varphi(v) - \varphi(h^{-1}(c)) \in C_0^\infty(\Omega), c \in \mathbb{R}\} \subset D(A)$, hence $\overline{D(A)} = X$.
To show that A is T-accretive, let $H_\epsilon \in C^1(\mathbb{R})$ with $0 \le H_\epsilon \le 1$, $H'_\epsilon \ge 0$ and $H_\epsilon(r) \to H_0(r)$ as $\epsilon \to 0+$ for all $r \in \mathbb{R}$, where $H_0(r) = 0$ for $r \le 0$ and $H_0(r) = 1$ for $r > 0$. Fix $\epsilon > 0$, let $(u,r) \in A(v,c)$ and $(\overline{u},\overline{r}) \in A(\overline{v},\overline{c})$. Then partial integration yields

$$\int_\Omega (u - \overline{u}) H_\epsilon(\varphi(v) - \varphi(\overline{v}))dx = -\int_\Omega (\Delta\varphi(v) - \Delta\varphi(\overline{v}))H_\epsilon(\varphi(v) - \varphi(\overline{v}))dx =$$

$$-\gamma \int_\Gamma (c - \overline{c} - (h(v) - h(\overline{v}))H_\epsilon(\varphi(v) - \varphi(\overline{v}))d\sigma + \int_\Omega |\nabla(\varphi(v) - \varphi(\overline{v}))|^2 H'_\epsilon(\varphi(v) - \varphi(\overline{v}))dx$$

$$\ge \gamma \int_\Gamma (h(v) - h(\overline{v}) - (c - \overline{c}))H_\epsilon(\varphi(v) - \varphi(\overline{v}))d\sigma.$$

Therefore $\epsilon \to 0+$ yields

$$\max \int_\Omega (u - \overline{u}) H(v - \overline{v})dx = \max \int_\Omega (u - \overline{u}) H(\varphi(v) - \varphi(\overline{v}))dx \ge$$

$$\gamma \int_\Gamma (h(v) - h(\overline{v}) - (c - \overline{c}))H_0(\varphi(v) - \varphi(\overline{v}))d\sigma.$$

This implies

$$\frac{1}{\gamma}\Big[\begin{pmatrix} v \\ c \end{pmatrix} - \begin{pmatrix} \overline{v} \\ \overline{c} \end{pmatrix}, \begin{pmatrix} u \\ r \end{pmatrix} - \begin{pmatrix} \overline{u} \\ \overline{r} \end{pmatrix}\Big]_+ \ge$$

$$\int_\Gamma (h(v) - h(\overline{v}) - (c - \overline{c}))H_0(\varphi(v) - \varphi(\overline{v}))d\sigma + \max\Big(H(c - \overline{c})\int_\Gamma (c - \overline{c} - (h(v) - h(\overline{v}))d\sigma\Big) \ge$$

$$\int_\Gamma (h(v) - h(\overline{v}))^+ d\sigma - (c - \overline{c})\int_\Gamma H_0(v - \overline{v})d\sigma + \sigma(\Gamma)(c - \overline{c})^+ - H_0(c - \overline{c})\int_\Gamma (h(v) - h(\overline{v}))d\sigma \ge 0.$$

Thus A is T-accretive, hence also accretive due to the particular Banach lattice X. Consequently, J_λ is order-preserving for all $\lambda > 0$, and the resolvent estimate stated above then follows from $(0,0) \in A(\overline{v},\overline{c})$ for $\overline{v} \equiv h^{-1}(\overline{c})$. For example, the choice of $\overline{v} \equiv \max\{|v^+|_\infty, h^{-1}(c)\}$ and $\overline{c} = \max\{h(|v^+|_\infty), c\}$ yields the second inequality in (21).

In order to show $R(I + \lambda A) = X$ for some $\lambda > 0$, let $(\overline{v},\overline{c}) \in X$ and $0 < \lambda < (\gamma\sigma(\Gamma))^{-1}$. To obtain a solution of

$$v - \lambda\Delta\varphi(v) = \overline{v} \text{ in } \Omega, \quad \frac{\partial\varphi(v)}{\partial\nu} = \gamma(c - h(v)) \text{ on } \Gamma, \quad c + \lambda\int_\Gamma \gamma(c - h(v))d\sigma = \overline{c}, \tag{22}$$

fix $c \in \mathbb{R}$ and let $\tilde{\varphi}(r) = \varphi(r + h^{-1}(c)) - \varphi(h^{-1}(c))$, $\beta(r) = \gamma(h(r + h^{-1}(c)) - c)$, $\overline{u} = \overline{v} - h^{-1}(c)$. Let A_0 denote the operator defined by (2) with $\tilde{\varphi}$ instead of φ. Since A_0 is m-accretive in $L^1(\Omega)$, there is $u \in D(A_0)$ such that

$$u - \lambda\Delta\tilde{\varphi}(u) = \overline{u} \text{ in } \Omega, \quad -\frac{\partial\tilde{\varphi}(u)}{\partial\nu} = \beta(u) \text{ on } \Gamma.$$

It is then easy to see that $v_c = u + h^{-1}(c)$ is a solution of

$$v - \lambda\Delta\varphi(v) = \overline{v} \;\text{ in } \Omega, \quad \frac{\partial\varphi(v)}{\partial\nu} = \gamma(c - h(v)) \;\text{ on } \Gamma.$$

Now consider solutions v_1 and v_2 of the latter equation for $c = c_1$ and $c = c_2$, respectively. Exploitation of

$$|v_1 - v_2|_1 = \lim_{\epsilon\to 0+} \int_\Omega (v_1 - v_2)s_\epsilon(\varphi(v_1) - \varphi(v_2))dx$$

where $s_\epsilon \in C^1(\mathbb{R})$ with $-1 \le s_\epsilon \le 1$, $s'_\epsilon \ge 0$ and $s_\epsilon(r) \to \operatorname{sgn}(r)$ as $\epsilon \to 0+$, partial integration and $\epsilon \to 0+$ yields

$$|v_1 - v_2|_1 \le \lambda\gamma\sigma(\Gamma)|c_1 - c_2|.$$

Define $\Psi : \mathbb{R} \to \mathbb{R}$ by $\Psi(c) = \overline{c} - \lambda \int_\Gamma \gamma(c - h(v_c))d\sigma$. Due to

$$\Psi(c) = \overline{c} - \lambda \int_\Omega \Delta\varphi(v_c)dx = \overline{c} + \int_\Omega (\overline{v} - v_c)dx,$$

the estimate above implies

$$|\Psi(c_1) - \Psi(c_2)| \le \lambda\gamma\sigma(\Gamma)|c_1 - c_2| \quad \text{for } c_1, c_2 \in \mathbb{R},$$

hence Ψ is a contraction for λ as chosen above. Therefore Ψ has a fixed point c for which (v_c, c) is a solution of (22).

It remains to prove the compactness property of the semigroup, where it suffices to consider $B \subset X$ bounded in $L^\infty(\Omega; \mathbb{R}_+) \times \mathbb{R}_+$ since $S(t)$ is nonexpansive, and $W \subset L^1(\Omega)$ for bounded Ω is weakly relatively compact iff

$$\sup_{w\in W} |w - w\chi_{\{|w|\le R\}}|_1 \to 0 \text{ as } R \to \infty.$$

For this purpose let us first show that $N_1(v, c) = |\Phi(v)|_1 + G(c)$ with $\Phi(r) = \int_0^r \varphi(s)ds$ and $G(c) = \int_0^c \varphi(h^{-1}(r))dr$ is a Lyapunov function for A. Let $(\overline{v}, \overline{c}) \in X$ and $(v, c) = J_\lambda(\overline{v}, \overline{c})$ for $\lambda > 0$. By convexity of $\Phi \ge 0$ and G it follows that

$$\begin{aligned}
N_1(\overline{v}, \overline{c}) &= \int_\Omega \Phi(v - \lambda\Delta\varphi(v))dx + G(c + \lambda \int_\Gamma \gamma(c - h(v))d\sigma) \\
&\ge \int_\Omega \Phi(v)dx - \lambda \int_\Omega \varphi(v)\Delta\varphi(v)dx + G(c) + \lambda\varphi(h^{-1}(c)) \int_\Gamma \gamma(c - h(v))d\sigma \\
&\ge N_1(v, c) + \lambda \int_\Omega |\nabla\varphi(v)|^2 dx + \lambda\gamma \int_\Gamma (\varphi(h^{-1}(c)) - \varphi(v))(c - h(v))d\sigma,
\end{aligned}$$

hence

$$N_1(v, c) + \lambda \int_\Omega |\nabla\varphi(v)|^2 dx \le N_1(\overline{v}, \overline{c}),$$

since φ and h are increasing. Application of Lemma 6.3 yields

$$|\Phi(v(t))|_1 + G(c(t)) + \int_0^t |\nabla\varphi(v(s))|_2^2 ds \le |\Phi(v_0)|_1 + G(c_0) \ \text{ for all } t > 0 \tag{23}$$

with $(v(\cdot), c(\cdot)) = S(\cdot)(v_0, c_0)$.

To deduce estimates on $|\nabla\varphi(v(t))|_2$ for every $t > 0$, we supplement N_1 by a second Lyapunov-like function. Fix $\alpha, \beta > 0$ such that $|v|_\infty \le \alpha$ and $c \le \beta$ for all $(v, c) \in B$. Let $a = \max\{\alpha, h^{-1}(\beta)\}$, $H(r) = \int_0^r h(\varphi^{-1}(s))ds$ and define $N_2 : D(N_2) \to \mathbb{R}$ by

$$N_2(t, v, c) = e^{-t}\Big(\frac{1}{2}|\nabla\varphi(v)|_2^2 + \gamma \int_\Gamma (H(\varphi(v)) - \varphi(v)c d\sigma\Big) \ \text{ on}$$

$$D(N_2) = \{(t, v, c) \in \mathbb{R}_+ \times X^+ : \varphi(v) \in W^{1,2}(\Omega), |v|_\infty \le a, c \le h(a)\}.$$

Let $(t, \overline{v}, \overline{c}) \in D(N_2)$ and $(v, c) = J_\lambda(\overline{v}, \overline{c})$ with $\lambda > 0$. Notice first that (21) yields $(v, c) \in X^+$ as well as $|v|_\infty \le a$ and $c \le h(a)$. Due to convexity of $H \ge 0$ we obtain

$$e^t[N_2(t + \lambda, v, c) - N_2(t, \overline{v}, \overline{c})] \le$$

$$\begin{aligned}
&\frac{1}{2}e^{-\lambda} \int_\Omega \langle\nabla(\varphi(v) - \varphi(\overline{v})), \nabla(\varphi(v) + \varphi(\overline{v}))\rangle dx - \frac{1 - e^{-\lambda}}{2}|\nabla\varphi(\overline{v})|_2^2 \\
&+ \gamma e^{-\lambda} \int_\Gamma (H(\varphi(v)) - H(\varphi(\overline{v})))d\sigma - \gamma e^{-\lambda} c \int_\Gamma \varphi(v)d\sigma + \gamma\overline{c} \int_\Gamma \varphi(\overline{v})d\sigma \\
&\le e^{-\lambda} \int_\Omega \langle\nabla(\varphi(v) - \varphi(\overline{v})), \nabla\varphi(v)\rangle dx - \frac{1 - e^{-\lambda}}{2}|\nabla\varphi(\overline{v})|_2^2 \\
&+ \gamma e^{-\lambda} \int_\Gamma (\varphi(v) - \varphi(\overline{v}))h(v)d\sigma - \gamma e^{-\lambda} c \int_\Gamma \varphi(v)d\sigma + \gamma\overline{c} \int_\Gamma \varphi(\overline{v})d\sigma.
\end{aligned}$$

Using partial integration, the resolvent equation and monotonicity of φ it follows that

$$e^t[N_2(t + \lambda, v, c) - N_2(t, \overline{v}, \overline{c})] \le \gamma(\overline{c} - e^{-\lambda}c) \int_\Gamma \varphi(\overline{v})d\sigma - \frac{1 - e^{-\lambda}}{2}|\nabla\varphi(\overline{v})|_2^2.$$

Exploitation of the resolvent equation for c together with $1 - e^{-\lambda} \le \lambda$ implies

$$e^t[N_2(t + \lambda, v, c) - N_2(t, \overline{v}, \overline{c})]$$

$$\begin{aligned}
&\le \lambda\gamma[\overline{c} + \gamma e^{-\lambda} \int_\Gamma (c - h(v))d\sigma] \int_\Gamma \varphi(\overline{v})d\sigma - \frac{1 - e^{-\lambda}}{2}|\nabla\varphi(\overline{v})|_2^2 \\
&\le \lambda\gamma[\overline{c} + \gamma\sigma(\Gamma)c] \int_\Gamma \varphi(\overline{v})d\sigma - \frac{1 - e^{-\lambda}}{2}|\nabla\varphi(\overline{v})|_2^2,
\end{aligned}$$

since $h(v) \ge 0$. By means of the inequalities incorporated into $D(N_2)$ this yields

$$e^t[N_2(t + \lambda, v, c) - N_2(t, \overline{v}, \overline{c})] \le \lambda\gamma\mu \int_\Gamma \varphi(\overline{v})d\sigma - \frac{1 - e^{-\lambda}}{2}|\nabla\varphi(\overline{v})|_2^2 \tag{24}$$

with $\mu = h(a)(1+\gamma\sigma(\Gamma))$.
Now recall that the trace operator is continuous from $W^{1,2}(\Omega)$ to $L^2(\Gamma)$. Since the latter is continuously embedded into $L^1(\Gamma)$, there exists $K>0$ such that

$$\int_\Gamma \varphi(\overline{v})d\sigma \le K\Big(|\varphi(\overline{v})|_2^2 + |\nabla\varphi(\overline{v})|_2^2\Big)^{1/2} \le K\Big(\sqrt{\lambda_n(\Omega)}|\varphi(\overline{v})|_\infty + |\nabla\varphi(\overline{v})|_2\Big),$$

hence

$$\int_\Gamma \varphi(\overline{v})d\sigma \le K\Big(\sqrt{\lambda_n(\Omega)}|\varphi(\overline{v})|_\infty + \kappa + \frac{1}{4\kappa}|\nabla\varphi(\overline{v})|_2^2\Big) \quad \text{for all } \kappa>0. \tag{25}$$

The choice of $\kappa = \gamma\mu K$, combined with (24) implies

$$N_2(t+\lambda, v, c) - N_2(t,\overline{v},\overline{c}) \le \lambda M e^{-t} \ \text{ with } M = \gamma\mu K\Big(\sqrt{\lambda_n(\Omega)}\varphi(a) + \gamma\mu K\Big)$$

for all small $\lambda>0$. Consequently, application of Lemma 6.3 shows that

$$t \to e^{-t}\Big(\frac{1}{2}|\nabla\varphi(v(t))|_2^2 + \gamma\int_\Gamma (H(\varphi(v(t))) - \varphi(v(t))c(t))d\sigma - M\Big)$$

is decreasing, where $(v(\cdot), c(\cdot)) = S(\cdot)(v_0, c_0)$. Given $0\le s\le t$, this obviously implies

$$e^{-t}\Big(\frac{1}{2}|\nabla\varphi(v(t))|_2^2 - \gamma c(t)\int_\Gamma \varphi(v(t))d\sigma - M\Big) \le e^{-s}\Big(\frac{1}{2}|\nabla\varphi(v(s))|_2^2 + \gamma\int_\Gamma H(\varphi(v(s)))d\sigma\Big).$$

For $v\in L^\infty(\Omega)$ with $0\le v\le a$ and $\varphi(v)\in W^{1,2}(\Omega)$ one easily sees that $H(\varphi(v))\in W^{1,2}(\Omega)$ with $|H(\varphi(v))|_{1,2} \le h(a)|\varphi(v)|_{1,2}$ and therefore

$$\int_\Gamma H(\varphi(v))d\sigma \le h(a)K\Big(\sqrt{\lambda_n(\Omega)}|\varphi(v)|_\infty + \kappa + \frac{1}{4\kappa}|\nabla\varphi(v)|_2^2\Big) \quad \text{for all } \kappa>0.$$

This estimate with $\kappa = \dfrac{\gamma h(a)K}{2}$ and (25) with $\kappa = \gamma c(t)K$ imply

$$e^{-t}\Big(\frac{1}{4}|\nabla\varphi(v(t))|_2^2 - \gamma c(t)K(\sqrt{\lambda_n(\Omega)}|\varphi(v(t))|_\infty + \gamma c(t)K) - M\Big)$$
$$\le e^{-s}\Big(|\nabla\varphi(v(s))|_2^2 + \gamma h(a)K(\sqrt{\lambda_n(\Omega)}|\varphi(v(s))|_\infty + \frac{1}{2}\gamma h(a)K)\Big).$$

Since $|\varphi(v(\cdot))|_\infty \le \varphi(a)$ and $c(t)\le h(a)$, integration of this inequality over $[0,t]$ together with (23) implies existence of $C_B>0$ such that

$$|\nabla\varphi(v)|_2^2 \le C_B\frac{e^t}{t} \quad \text{for all } (v,c)\in S(t)B;$$

notice that $S(\tau)(v_0,c_0)\in D(N_2)$ for $(v_0,c_0)\in B$ and arbitrarily small $\tau>0$ by (23). Therefore $\{\varphi(v) : (v,c)\in S(t)B\}$ is bounded in $W^{1,2}(\Omega)$, hence relatively compact in $L^2(\Omega)$ for all $t>0$. This evidently yields relative compactness of $S(t)B$ in $L^1(\Omega)\times\mathbb{R}$. □

Now we are able to prove existence of T-periodic solutions of (17).

Theorem 6.4 *Let $\Omega \subset \mathbb{R}^n$ be open bounded with C^2-boundary Γ. For every $k = 1, \ldots, m$ let $\gamma_k > 0$, $\varphi_k : \mathbb{R} \to \mathbb{R}$ be continuous, strictly increasing with $\varphi_k(0) = 0$ and $h_k : \mathbb{R} \to \mathbb{R}$ be continuous, strictly increasing and onto with $h_k(0) = 0$. Let $r : \mathbb{R} \times \Omega \times \mathbb{R}^m_+ \to \mathbb{R}^m$ and $R : \mathbb{R} \times \mathbb{R}^m_+ \to \mathbb{R}^m$ be measurable and T-periodic with respect to t and continuous in the other variables. In addition, suppose that there exist $\overline{c}, \overline{y} \in \mathbb{R}^m_+$ with $\overline{c}_k = h_k(\overline{y}_k)$ and $d \in L^1([0,T])$ such that*

$$r_k(t,x,y) \geq 0 \text{ for } 0 \leq y \leq \overline{y} \text{ with } y_k = 0, \quad R_k(t,c) \geq 0 \text{ for } 0 \leq c \leq \overline{c} \text{ with } c_k = 0,$$
$$r_k(t,x,y) \leq 0 \text{ for } 0 \leq y \leq \overline{y} \text{ with } y_k = \overline{y}_k, \quad R_k(t,c) \leq 0 \text{ for } 0 \leq c \leq \overline{c} \text{ with } c_k = \overline{c}_k,$$
$$|r_k(t,x,y)| \leq d(t) \text{ on } [0,T] \times \Omega \times [0,\overline{y}], \quad |R_k(t,c)| \leq d(t) \text{ on } [0,T] \times [0,\overline{c}].$$

Then the reaction-diffusion system (17), considered as the abstract problem (19), has a nonnegative T-periodic mild solution. In case r and R are independent of t, system (17) admits a nonnegative stationary solution.

Proof. Let $J = [0,T]$ and $X = (L^1(\Omega) \times \mathbb{R})^m$ with $|u| = |u_1| + \ldots + |u_m|$ and $|u_k| = |v_k|_1 + |c_k|$ for $u_k = (v_k, c_k)$, equipped with the partial ordering $u \leq \overline{u}$ if $v_k \leq \overline{v}_k$ and $c_k \leq \overline{c}_k$ for all $k = 1, \ldots, m$. Define A by

$$Au = (A_1 u_1, \ldots, A_m u_m) \text{ on } D(A) = D(A_1) \times \ldots \times D(A_m)$$

with operators A_k given by (18) where φ, h and γ are replaced by φ_k, h_k and γ_k, respectively. Then A is m-accretive in X since all A_k have this property in $L^1(\Omega) \times \mathbb{R}$ by Lemma 6.4.
Let $K = K_1 \times \ldots \times K_m \subset X$ with

$$K_k = \{(v,c) \in L^1(\Omega) \times \mathbb{R} : 0 \leq v \leq \overline{y}_k \text{ a.e. on } \Omega,\ 0 \leq c \leq \overline{c}_k\}.$$

As a consequence of (21) in Lemma 6.4 the resolvents satisfy $(I + \lambda A_k)^{-1} K_k \subset K_k$ for all $\lambda > 0$ and $S_k(t) K_k$ is relatively compact in $L^1(\Omega) \times \mathbb{R}$ for all $t > 0$. Therefore $J_\lambda K \subset K$ and $S(t)K$ is relatively compact in X for all $\lambda, t > 0$.
Define $f : J \times K \to X$ by $f_k(t,u)(x) = (r_k(t,x,v_1(x), \ldots, v_m(x)), R_k(t, c_1, \ldots, c_m))$. It is easy to check that $|f(t,u)| \leq m(1 + \lambda_n(\Omega)) d(t)$ on $J \times K$, and the Carathéodory property of r and R is inherited by f. Moreover, given $t \in J$ and $u \in K$, the conditions on r_k and R_k imply

$$\frac{1}{h} \rho(v_k(x) + h r_k(t,x,v_1(x), \ldots, v_m(x)), [0, \overline{y}_k]) \to 0 \text{ a.e. on } \Omega \text{ as } h \to 0+,$$
$$\frac{1}{h} \rho(c_k + h R_k(t, c_1, \ldots, c_m), [0, \overline{c}_k]) \to 0 \text{ as } h \to 0+.$$

The dominated convergence theorem implies $f_k(t,u) \in T_{K_k}(u_k)$ for all $k = 1, \ldots, m$ due to boundedness of r_k, R_k for fixed t, hence $f(t,u) \in T_K(u)$ on $J \times K$. Consequently, Theorem 5.3 yields a T-periodic mild solution $u(\cdot)$ of (19) which is nonnegative since $u(t) \in K$ on J. In the autonomous case, Corollary 5.1 provides existence of a stationary solution $u \in K$. $\square$

Under the following natural assumptions, Theorem 6.4 applies with $\bar{c} = (|c_A^f|_\infty, |c_B^f|_\infty)$ to the special reaction-diffusion system (16).

Corollary 6.1 *Let $\Omega \subset \mathbb{R}^n$ be open bounded with C^2-boundary Γ. Let $\varphi_A, \varphi_B : \mathbb{R} \to \mathbb{R}$ be continuous, strictly increasing with $\varphi_A(0) = \varphi_B(0) = 0$ and $h_A, h_B : \mathbb{R} \to \mathbb{R}$ be continuous, strictly increasing and onto with $h_A(0) = h_B(0) = 0$. Suppose that $c_A^f, c_B^f \in L^\infty(\mathbb{R}; \mathbb{R}_+)$ are T-periodic and $r : \mathbb{R}_+^2 \to \mathbb{R}_+$ is continuous such that $r(c_A, c_B) = 0$ if $c_A c_B = 0$.*
Then (16) admits a nonnegative T-periodic mild solution, and a stationary one if the feeds c_A^f, c_B^f are constant. If the feeds do not vanish, this solution is nontrivial.

6.4 Remarks

Remark 6.1 The model problem (4) with $m = 2$ and autonomous g has been considered in Maddalena [77], but with Dirichlet boundary conditions replaced by the mixed boundary conditions $\frac{\partial \varphi_k}{\partial \nu}(u_k) + \alpha_k \varphi_k(u_k) = 0$ on $(0, \infty) \times \Gamma$, where the α_k are sufficiently smooth nonnegative functions. In this paper existence of a global weak solution of (4) for $u_0 \in L^\infty(\Omega; \mathbb{R}_+^2)$ is obtained in the following situation: either $\varphi_k \equiv 0$ or $\varphi_k(0) = \varphi_k'(0) = 0$ and $\varphi_k(r), \varphi_k'(r), \varphi_k''(r) > 0$ for $r > 0$. Furthermore g is assumed to be smooth and quasimonotone with respect to $\mathbb{R}_+^2$ with $g(0) = 0$ such that for every $y \in \mathbb{R}_+^2$ there is $\overline{y} \geq y$ with $g(\overline{y}) \leq 0$. Notice that the latter two assumptions on g imply positive invariance of $[0, \overline{y}] \subset \mathbb{R}_+^2$ for $y' = g(y)$. If this holds then (4) admits a global mild solution for every $u_0 \in L^\infty(\Omega; \mathbb{R}_+^m)$ by Theorem 6.1 if the φ_k are continuous, strictly increasing with $\varphi_k(0) = 0$. Let us also note that, under the above assumptions on g, no compactness property of the semigroup corresponding to the diffusion terms is needed, since g is Lipschitz on every rectangle $[0, \overline{y}]$. Hence Theorem 4.1 yields a global mild solution for every $u_0 \in L^\infty(\Omega; \mathbb{R}_+^m)$, even if some of the φ_k vanish and the boundary condition for the remaining components is replaced by $-\partial_\nu \varphi_k(u_k) \in \beta_k(u_k)$ where β_k are maximal monotone graphs in $\mathbb{R}$.

Remark 6.2 It has been mentioned above that parabolic problems with discontinuous nonlinearities arise in certain limiting cases, and then a natural question is whether solutions of the limit problem are unique. Of course one cannot expect unique solvability in general, unless the nonlinearity is of dissipative type, but one may look for conditions that guarantee uniqueness for particular initial values. This is done in Feireisl/Norbury [53], where the parabolic inclusion

$$\begin{aligned} &u_t \in u_{xx} + f(u) + \lambda H(u-1) && \text{for } t > 0, x \in (0, \pi) \\ &u(t,0) = u(t,\pi) = 0 && \text{for } t > 0 \\ &u(0,x) = u_0(x) && \text{for } x \in (0, \pi) \end{aligned} \tag{26}$$

has been studied; here $f : \mathbb{R} \to \mathbb{R}_+$ is nondecreasing and Lipschitz with $f(r) = 0$ for $r \leq 1$, and $\lambda > 0$. In this paper local uniqueness is obtained for a certain set of initial values

$u_0 \in C^1([0,\pi])$, where the main assumption is given by $u_0'(x) \neq 0$ on $\{x \in [0,\pi] : u_0(x) = 1\}$. Furthermore, the authors also provide an example for nonuniqueness in case $f = 0$.
Let us note that (26) can be rewritten in such a way that it becomes a special case (with $k \equiv 1$) of

$$\begin{aligned} &u_t \in (k(u)u_x)_x + \lambda f(u)H(u-1) && \text{for } t>0, x \in (-1,1) \\ &u(t,-1) = u(t,1) = 0 && \text{for } t>0 \\ &u(0,x) = u_0(x) && \text{for } x \in (-1,1). \end{aligned} \tag{27}$$

Based on the considerations given in Norbury/Stuart [88], the latter has been proposed in Stuart [104] as a model for combustion in porous media in the limiting case of large activation energy. The n-dimensional version of this model problem has been studied in Gianni [59] (under certain assumptions that especially imply uniform parabolicity), in particular with respect to regularity properties of the free boundary $\{u = 1\}$.

Remark 6.3 Let us provide some additional information concerning global existence for semilinear reaction-diffusion systems of the type

$$\frac{\partial u}{\partial t} = D\Delta u + g(u) \text{ in } (0,\infty)\times\Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } (0,\infty)\times\Gamma, \quad u(0,\cdot) = u_0 \text{ in } \Omega, \tag{28}$$

where $u = (u_1, \ldots, u_m)$, $D = \operatorname{diag}(d_1, \ldots, d_m)$ with $d_k > 0$ and $\Omega \subset \mathbb{R}^n$ is open bounded with smooth boundary Γ. If the reaction term refers to a concrete systems of chemical reactions then g will be quasi-positive, and in practice it is almost always possible to find some $e \in \overset{\circ}{\mathbb{R}}{}^n_+$ such that $\langle g(y), e\rangle \leq 0$ on $\mathbb{R}^m_+$; observe that the latter is closely related to mass conservation since it means that the quantity $e_1y_1 + \cdots + e_my_m$ is decreasing. In this case every set of the form $C_s = \{y \in \mathbb{R}^m_+ : \langle y, e\rangle \leq s\}$ is compact convex and positively invariant for the ordinary differential equation $y' = g(y)$, associated with (28). Consequently, application of invariance techniques yields global existence of solutions with initial value $u_0 \in L^\infty(\Omega; \mathbb{R}^m_+)$ if $D = dI$ (with $d > 0$), but the latter is not realistic in concrete applications. For general D it has been mentioned above that the same approach works if $y' = g(y)$ admits positively invariant rectangles (or, more generally, positively invariant tubes with rectangular values). This invariance approach to obtain global existence for semilinear reaction-diffusion systems is summarized in Martin [79]; the results given there are closely related to Theorem 6.1 and Proposition 6.1.

Unfortunately, application of this technique in case of general D requires very strong assumptions on the underlying systems of chemical reactions. Let us illustrate this be means of a few concrete examples. In the simplest case $m = 2$ a single irreversible reaction $\alpha A + \beta B \to P$ typically leads to a reaction term given by the so-called Freundlichs kinetics, namely

$$g(y) = (-\alpha k y_1^\alpha y_2^\beta, -\beta k y_1^\alpha y_2^\beta) \quad \text{with } \alpha, \beta, k > 0;$$

here the stoichiometric coefficients α and β represent the order of the reaction with respect to A and B, respectively, and k is the rate constant. In this situation every rectangle $[0, \overline{y}] \subset \mathbb{R}^2_+$

is obviously positively invariant, hence global existence for L^∞-initial values holds. In case of a mixed order reversible reaction $\alpha A \rightleftharpoons \beta B$ one has

$$g(y) = (\alpha(k_2 y_2^\beta - k_1 y_1^\alpha), \beta(k_1 y_1^\alpha - k_2 y_2^\beta)) \quad \text{with } \alpha, \beta, k_1, k_2 > 0.$$

Here g is quasimonotone with $g(k_1^{-1/\alpha} r^\beta, k_2^{-1/\beta} r^\alpha) = 0$ for all $r > 0$, and therefore (8) and in particular (28) admit global solutions for every initial value in $L^\infty(\Omega; \mathbb{R}_+^2)$ by Proposition 6.1. The situation is different for a single reversible reaction of type $\alpha A + \beta B \rightleftharpoons P$, even if $\alpha = \beta = 1$. In this special case the kinetics is usually described by means of $g(y) = k\nu r(y)$ with $k > 0$, $\nu = (-1, -1, 1)^T$ and $r(y) = y_1 y_2 - \kappa y_3$ with $\kappa > 0$. Here $y_1 + y_2 + 2y_3$ is a conserved quantity which yields convex invariant sets of triangular structure, and the invariance approach is not applicable for different diffusion coefficients. This particular system has been studied in Rothe [99] where global existence has been obtained in case $n \leq 5$. The case $\beta > 1$ is more difficult and considerable effort has been made to establish global existence for the related model problem (28) with $m = 2$ and $g(y) = (-y_1 y_2^\beta, y_1 y_2^\beta)$. For this system global existence for L^∞-initial values has been obtained in Alikakos [1] in case $1 \leq \beta \leq 1 + n/2$, and in Masuda [82] for arbitrary $\beta \geq 1$. In Hollis/Martin/Pierre [69] this result is extended to a more general class of reaction terms $g : \mathbb{R}_+^2 \to \mathbb{R}^2$ in case the corresponding system has similar structural properties. More precisely, global existence for system (28) with $m = 2$ is proven under the following assumptions: $g : \mathbb{R}_+^2 \to \mathbb{R}^2$ is C^1 and quasi-positive such that $g_1(y) + g_2(y) \leq \varphi(r)$ for $y \in \mathbb{R}_+^2$ with $y_1 \leq r$ and $|g_2(y)| \leq \psi(r)(1 + y_2)^\gamma$ for $y \in \mathbb{R}_+^2$ with $y_2 \leq r$, where $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\gamma \geq 1$, and there is a continuous $N : \mathbb{R}_+ \to \mathbb{R}_+$ such that $|u_1(t, \cdot)|_\infty \leq N(t)$ on the maximal interval of existence; in fact this result is obtained for time-dependent g.

Further extensions to systems with more than two components have been obtained in Morgan [86], allowing in particular for systems with quasi-positive, locally Lipschitz g of at most polynomial growth such that $Mg(y) \leq Ly + a$ with $a \in \mathbb{R}^m$, $L \in \mathbb{R}^{m \times m}$ and a nonnegative, invertible, lower triangular $m \times m$-matrix M. This assumptions are satisfies for a large class of concrete systems; observe for example that $g(y) = k\nu r(y)$ from above, corresponding to $A + B \rightleftharpoons P$, is admissible by the choice of

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 0 & 0 & k\kappa \\ 0 & 0 & k\kappa \\ 0 & 0 & 0 \end{pmatrix}.$$

Nevertheless, the practically important case of $g(y) = k\nu r(y)$ with $\nu = (-1, -1, 1, 1)^T$ and $r(y) = y_1 y_2 - \kappa y_3 y_4$ (with $\kappa > 0$), corresponding to $A + B \rightleftharpoons P + Q$, is not covered. Actually, global existence for this particular system seems to be an open problem.

Additional information and a survey of different techniques to establish global existence can be found in Martin/Pierre [81].

Remark 6.4 In Diaz/Vrabie [46] the authors consider (12) in case $m = 2$, φ_k continuous increasing with $\varphi_k(0) = 0$ and $G(y) = G_1(y) \times G_2(y)$ where the $G_k : \mathbb{R}^2 \to 2^{\mathbb{R}} \setminus \emptyset$ are usc with compact convex values. In Theorem 2.1 of this paper existence of a local (weak) solution of (12) with $u_0 \in L^\infty(\Omega)^2$ is obtained if the φ_k are also strictly increasing. The same result provides the existence of a global solution of (12) if, in addition, there exist $R, c > 0$ such that $|y|_\infty > R$ implies $\max\{y_1 z_1, y_2 z_2\} \leq 0$ for some $z \in G(y)$ or $|z|_\infty \leq c(1 + |y|_\infty)$ for all $z \in G(y)$, where $|y|_\infty = \max\{|y_1|, |y_2|\}$. If only φ_1 is strictly increasing, the same assertions hold under strong additional assumptions on G_2; see Theorem 2.2 in [46].

Let us note in passing that for continuous, strictly increasing φ_k problem (12) has a global solution for $u_0 \in L^\infty(\Omega)^m$ if $G : \mathbb{R}^m \to 2^{\mathbb{R}^m} \setminus \emptyset$ is usc with compact convex values such that, for some $R, c > 0$, $|y|_\infty > R$ implies $\max\{y_k z_k : k = 1, \ldots, m\} \leq 0$ or $|z|_\infty \leq c(1 + |y|_\infty)$ for some $z \in G(y)$. To see this, let $r(\cdot)$ be the solution of $r' = c(1 + r) + 1$ on $\mathbb{R}_+$ with $r(0) > \max\{R, |u_{0,k}|_\infty\}$, and $C(t) = [-r(t), r(t)]^m$. Then the above condition on G implies $G(y) \cap T_C(t, y) \neq \emptyset$ on $\mathrm{gr}\,(C)$, hence a simple modification of Theorem 6.3 (where we have concentrated on the case $C(t) = [0, \overline{y}(t)] \subset \mathbb{R}^m_+$ to get nonnegative solutions) yields a global mild solution u of (12).

§7 Instantaneous Irreversible Reactions

We consider chemically reacting systems involving fast irreversible reactions with additional mass transport due to diffusion or macroscopic convection. A main purpose of the present paragraph is to show that ordinary differential equations with discontinuous right-hand side as well as problems with nonlinear diffusion arise naturally if such systems are considered in the limiting case of instantaneous reactions. In special situations this leads to evolution problems governed by m-accretive operators with multivalued perturbations of usc type.

7.1 Reactions with macroscopic convection

In this section we concentrate on a rather simple but instructive example of two concurring irreversible reactions with macroscopic convection. This example illustrates how nonlinear semigroup theory can be applied to obtain convergence of solutions to the solution of a certain limit problem, where the latter turns out to be a differential inclusion that corresponds to an ordinary differential equation with discontinuous right-hand side.

Example 7.1 Consider two concurring chemical reactions $C_1+C_2 \to P_1$, $C_1+C_3 \to P_2$ which are performed inside a continuously stirred tank reactor (CSTR), i.e. the reacting species are fed into the reactor via a carrying liquid, reaction takes place inside the reactor within the liquid which is ideally mixed, and products as well as remaining educts are removed through outlets. Since P_1 and P_2 only appear as products, it suffices to consider the time-evolution of the vector $c = (c_1, c_2, c_3)$ of the concentrations of the species C_i. We suppose that the rate functions are given by so-called mass-action kinetics (see Chapter 4 in Erdi/Toth [49]), and in this case the rate function for a reaction of type $A + B \to P$ is given by $r(c_A, c_B) = kc_Ac_B$ with rate constant $k > 0$; here the basic idea is that the reaction rate is proportional to the probability of a "collision" of the involved particles, and that this probability is in turn proportional to the product of the corresponding concentrations. This leads to the system

$$\begin{aligned}
\dot{c}_1 &= f_1(c) - k_1c_1c_2 - k_2c_1c_3 \\
\dot{c}_2 &= f_2(c) - k_1c_1c_2 \\
\dot{c}_3 &= f_3(c) - k_2c_1c_3,
\end{aligned}$$

where the f_i model in- and outflow as well as additional slow reactions. If further species $C_4, C_5, \ldots$ are involved in the slow reactions then the system of course has to be enlarged. This does not affect the subsequent considerations, but for simplicity we continue with this smaller system.

We obtain convergence of solutions as $k_1, k_2 \to \infty$, where the quotient k_1/k_2 is kept constant since this ratio determines how the different species contribute to the different reactions. Consequently, we consider the limiting process $k \to \infty$ for the initial value problem

$$\dot{c} = f(c) - kg(c) \quad \text{on } \mathbb{R}_+, \quad c(0) = c_0, \tag{1}$$

where

$$g(c) = (c_1c_2 + \lambda c_1c_3, c_1c_2, \lambda c_1c_3) \ \text{ with } \lambda > 0.$$

Concerning f we assume that $f : \mathbb{R}^3_+ \to \mathbb{R}^3$ is locally Lipschitz and quasi-positive such that

$$\langle e, f(c)\rangle \leq a(1 + |c|) \ \text{ on } \mathbb{R}^3_+ \text{ with } a \geq 0 \text{ and } e \in \overset{\circ}{\mathbb{R}^3_+}; \tag{2}$$

these assumptions are reasonable as will be explained in a more general setting in §8.2 below.

Our aim is to prove that

$$c^k \to c^\infty \ \text{ in } C([\delta, T]; \mathbb{R}^3) \text{ as } k \to \infty \text{ for all } 0 < \delta < T, \tag{3}$$

where c^∞ is the unique solution of the limit inclusion

$$\dot{c} \in F(c) \ \text{ on } \mathbb{R}_+, \ \ c(0) = c_0^\infty. \tag{4}$$

Here the right-hand side F is defined on $D = \{c \in \mathbb{R}^3_+ : c_1c_2 = c_1c_3 = 0\}$ by means of

$$F(c) = \begin{cases} \{(f_1(c) - f_2(c) - f_3(c), 0, 0)\} & \text{if } c_1 > 0, c_2 = c_3 = 0 \\ \{(0, f_2(c) - \dfrac{c_2}{c_2 + \lambda c_3} f_1(c), f_3(c) - \dfrac{\lambda c_3}{c_2 + \lambda c_3} f_1(c))\} & \text{if } c_1 = 0, c_2 + c_3 > 0 \\ \operatorname{conv}\Big(\{(f_1(0) - f_2(0) - f_3(0), 0, 0), & \\ \quad (0, f_2(0) - f_1(0), f_3(0)), & \\ \quad (0, f_2(0), f_3(0) - f_1(0))\}\Big) & \text{if } c_1 = c_2 = c_3 = 0. \end{cases} \tag{5}$$

The initial value c_0^∞ in (4) is determined by the equations

$$\begin{array}{ll} c_{0,1}^\infty = c_{0,1} - c_{0,2} - c_{0,3}, \ \ c_{0,2}^\infty = c_{0,3}^\infty = 0 & \text{if } c_{0,1} \geq c_{0,2} + c_{0,3} \\ c_{0,1}^\infty = 0, \ \ c_{0,2}^\infty + c_{0,3}^\infty = c_{0,2} + c_{0,3} - c_{0,1}, \ \ (c_{0,2}^\infty)^\lambda c_{0,3} = (c_{0,2})^\lambda c_{0,3}^\infty & \text{if } c_{0,1} < c_{0,2} + c_{0,3}. \end{array} \tag{6}$$

First of all, existence of a unique local solution $c^k(\cdot)$ of (1) is a consequence of Corollary 2.1 since the right-hand side in (1) is locally Lipschitz and quasi-positive. Due to (2) we evidently obtain a priori bounds for $\langle e, c^k(t)\rangle$, uniformly in $k > 0$. Hence $c^k(\cdot)$ exists on all of $\mathbb{R}_+$ and satisfies $|c^k(t)| \leq M$ on $[0, T]$ for every $k > 0$ with some $M = M(c_0, T)$.

To prove (3) it suffices to consider problem (1) for fixed c_0 and on $[0, T]$ instead of $\mathbb{R}_+$. Hence we may assume that f is bounded and Lipschitz on $\mathbb{R}^3_+$, possibly after a modification of f outside of the set $\{c \in \mathbb{R}^3_+ : |c| \leq M(c_0, T)\}$. Let the norm on $\mathbb{R}^3$ be given by $|\cdot|_1$ and let $[\cdot, \cdot]$ denote the corresponding bracket, i.e. $[c, z] = \sum_{i=1}^{3} \max(z_i \operatorname{Sgn}(c_i))$. Then

$$[c - \overline{c}, g(c) - g(\overline{c})] \geq \sum_{i=1}^{3} (g_i(c) - g_i(\overline{c})) \ \operatorname{sgn}(c_i - \overline{c}_i)$$

$$= (c_1c_2 - \overline{c}_1\overline{c}_2)(\operatorname{sgn}(c_1 - \overline{c}_1) + \operatorname{sgn}(c_2 - \overline{c}_2))$$
$$+\lambda(c_1c_3 - \overline{c}_1\overline{c}_3)(\operatorname{sgn}(c_1 - \overline{c}_1) + \operatorname{sgn}(c_3 - \overline{c}_3)) \geq 0$$

for all $c, \overline{c} \in \mathbb{R}^3_+$, hence g is accretive. Therefore $A_k(c) = kg(c) - f(c)$ on $\mathbb{R}^3_+$ defines an ω-accretive operator, where $\omega > 0$ is a Lipschitz constant for f. We claim that $-F \subset A_\infty := \liminf_{k\to\infty} A_k$, which requires computation of A_∞. Since f is continuous we have $A_\infty = -f + B_\infty$, where $B_\infty = \liminf_{k\to\infty} B_k$ with $B_k(c) = kg(c)$ on $\mathbb{R}^3_+$. Let $(c, z) \in B_\infty$, i.e. $c = \lim_{k\to\infty} c^k$ and $z = \lim_{k\to\infty} kg(c^k)$ with $(c^k)_{k>0} \subset \mathbb{R}^3_+$. In particular this implies $g(c) = 0$, hence $D(B_\infty) = D(A_\infty) = D$. Now we only consider the case $c \in D$ with $c_1 > 0$ in more detail. Then $kg(c^k) \to z$ implies $kc_2^k \to a$, $kc_3^k \to b$ for some $a, b \geq 0$, hence $z = (\alpha + \beta, \alpha, \beta)$ with certain $\alpha, \beta \geq 0$. Conversely, if $c \in D$ with $c_1 > 0$ and $\alpha, \beta \geq 0$ are given then the choice of $c^k = (c_1, \alpha/(kc_1), \beta/(kc_1))$ for $k > 0$ shows that $z = (\alpha + \beta, \alpha, \beta)$ belongs to $B_\infty(c)$. By means of similar computations in the case $c_2 + c_3 > 0$, respectively $c = 0$, we obtain

$$A_\infty(c) = -f(c) + \begin{cases} \{\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} : \alpha, \beta \geq 0\} & \text{if } c_1 \geq 0, c_2 = c_3 = 0 \\ \{\alpha c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha\lambda c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} : \alpha \geq 0\} & \text{if } c_1 = 0, c_2 + c_3 > 0. \end{cases}$$

Then $-F \subset A_\infty$ follows directly from the definition of F. At this point some comments about the relation between A_∞ and F are in order. In view of Theorem 1.4 a natural candidate for the limit problem is

$$\dot{c} + A_\infty c \ni 0 \ \text{ on } \mathbb{R}_+, \ \ c(0) = c_0, \tag{7}$$

but this inclusion is not appropriate for practical purposes since A_∞ has unbounded values. On the other hand, every mild solution c of (7) for $c_0^\infty \in D$ is Lipschitz continuous since $D = D(A_\infty)$, hence c is a strong solution. Consequently c satisfies $\dot{c}(t) \in -A_\infty(c(t))$ a.e. on $\mathbb{R}_+$ and also $\dot{c}(t) \in T_D(c(t))$ a.e. on $\mathbb{R}_+$. Therefore F in (5) is defined as $F(c) = -A_\infty(c) \cap T_D(c)$ on $D\backslash\{0\}$. The value of F at the point $c = 0$ of discontinuity is obtained by usc regularization, i.e. $F(0) = \bigcap_{\delta>0} \overline{\operatorname{conv}} F(B_\delta(c) \cap D)$.

Now observe that F satisfies $F(0) \cap T_D(0) \neq \emptyset$ since, depending on the sign of $f_1(0) - f_2(0) - f_3(0)$, the set $F(0)$ intersects $\mathbb{R}_+ \times \{(0,0)\}$ or $\{0\} \times \mathbb{R}^2_+$. Hence $F : D \to 2^{\mathbb{R}^3} \setminus \emptyset$ is usc and bounded with compact convex values such that $F(c) \cap T_D(c) \neq \emptyset$ on D. Consequently, initial value problem (4) admits a strong solution for every $c_0^\infty \in D$ as a consequence of Theorem 2.1. Moreover, this solution is unique since F is ω-dissipative due to $-F \subset A_\infty$. Then application of Theorem 1.4 shows that

$$(c_0^k) \subset \mathbb{R}^3_+ \text{ with } c_0^k \to c_0^\infty \in D \text{ implies } c^k(\cdot; c_0^k) \to c^\infty(\cdot; c_0^\infty) \text{ in } C([0,T]; \mathbb{R}^3), \tag{8}$$

where $c^\infty(\cdot;c_0^\infty)$ denotes the solution of (4). In case $c_0 \in D$ this yields the convergence announced in (3); notice that $c_0 = c_0^\infty$ then. Given an arbitrary initial value $c_0 \in \mathbb{R}_+^3$ and $0 < \delta < T$, it remains to show that

$$c^k(\cdot;c_0) \to c^\infty(\cdot;c_0^\infty) \text{ in } C([\delta,T];\mathbb{R}^3) \text{ as } k \to \infty,$$

and the latter follows from (8) if $c^k(\delta;c_0) \to c^\infty(\delta;c_0^\infty)$ holds. Consider $z^k(t) = c^k(t/k;c_0)$ on $\mathbb{R}_+$. Evidently $z^k(\cdot)$ is the solution of

$$\dot z^k = \frac{1}{k} f(z^k) - g(z^k) \text{ on } \mathbb{R}_+, \quad z^k(0) = c_0,$$

hence $z^k(t) \to z(t)$ uniformly on bounded subsets of $\mathbb{R}_+$, where $z(\cdot)$ is the solution of

$$\dot z = -g(z) \text{ on } \mathbb{R}_+, \quad z(0) = c_0.$$

Since all components of $z(\cdot)$ are decreasing and nonnegative it follows that $\hat z := \lim\limits_{t\to\infty} z(t)$ exists. Suppose for the moment that $\hat z = c_0^\infty$. Then, given $\epsilon > 0$, there is $\sigma > 0$ such that $|z(\sigma) - c_0^\infty|_1 \le \epsilon$. Therefore $|z^k(\sigma) - c_0^\infty|_1 \le 2\epsilon$, i.e. $|c^k(\sigma/k;c_0) - c_0^\infty|_1 \le 2\epsilon$ for all $k \ge k_\epsilon$. Since all A_k are ω-accretive this implies

$$|c^k(\delta;c_0) - c^k(\delta - \sigma/k; c_0^\infty)|_1 \le |c^k(\sigma/k;c_0) - c_0^\infty|_1 e^{\omega(\delta-\sigma/k)} \le 2\epsilon e^{\omega\delta} \text{ for all large } k.$$

Hence $c^k(\cdot;c_0^\infty) \to c^\infty(\cdot;c_0^\infty)$ in $C([0,\delta];\mathbb{R}^3)$ by (8) yields

$$|c^k(\delta;c_0) - c^\infty(\delta;c_0^\infty)|_1 \le 2\epsilon e^{\omega\delta} + \epsilon \text{ for all large } k,$$

and therefore $c^k(\delta;c_0) \to c^\infty(\delta;c_0^\infty)$ as $k \to \infty$.

It remains to show $\hat z = c_0^\infty$. Evidently $\hat z_i \ge 0$ for $i = 1,2,3$ and $g(\hat z) = 0$, hence $\hat z_1 \hat z_2 = \hat z_1 \hat z_3 = 0$. Due to the special structure of g we also have $\hat z_1 - \hat z_2 - \hat z_3 = c_{0,1} - c_{0,2} - c_{0,3}$. Furthermore $z_2(t) = c_{0,2} \exp\Big(-\int_0^t z_1(s)ds\Big)$ and $z_3(t) = c_{0,3} \exp\Big(-\lambda \int_0^t z_1(s)ds\Big)$ imply $\hat z_2^\lambda c_{0,3} = \hat z_2\, c_{0,2}^\lambda$. Therefore $\hat z$ is a solution of the equations in (6), hence $\hat z = c_0^\infty$. $\diamond$

Let us note in passing that the convergence result given in Example 7.1 can be extended to the case of $m > 2$ fast irreversible reactions if at most two educts are involved in each fast reaction and all stoichiometric coefficients are equal. Here the first assumption is indispensible in order to obtain accretivity of the fast reaction term in (a weighted) l^1-norm; recall that the corresponding estimate in Example 7.1 relies on the fact that the sum $\operatorname{sgn}(c_1 - \overline{c}_1) + \operatorname{sgn}(c_2 - \overline{c}_2)$ vanishes on the set where no information on the sign of $c_1c_2 - \overline{c}_1\overline{c}_2$ is available. While this condition is often satisfied if elementary reactions are considered, the second assumption above is a severe restriction.

The occurence of discontinuous right-hand sides in the limiting case of instantaneous irreversible reactions is another motivation to study multivalued perturbations of m-accretive evolution problems, if (1) is only part of a larger ode/pde-system. Let us illustrate this point by means of the following example. Suppose that a heterogeneously catalysed reaction $A+B \to P$ is performed, that takes place inside porous catalytic pellets which are suspended in the bulk phase of a CSTR; cf. the explanations given in §6.3. Assume that A itself is formed by a preceding reaction $C+D \to A$ inside the bulk phase, while B is directly fed into the reactor and is accompanied by a further species E of impurities, say. Now if E in turn reacts with C then two concurring reactions $C+D \to A$ and $C+E \to Q$ occur, and we end up with the following mathematical model for the overall process.

$$\begin{aligned}
\partial_t u_A &= \Delta\varphi_A(u_A) - r(u_A,u_B) \text{ in } \Omega, \quad \partial_\nu \varphi_A(u_A) = \gamma_A(c_A - h_A(u_A)) \text{ on } \Gamma \\
\partial_t u_B &= \Delta\varphi_B(u_B) - r(u_A,u_B) \text{ in } \Omega, \quad \partial_\nu \varphi_B(u_B) = \gamma_B(c_B - h_B(u_B)) \text{ on } \Gamma \\
\dot{c}_A &= f_A(c) + kc_C c_D - \int_\Gamma \gamma_A(c_A - h_A(u_A))\, d\sigma \\
\dot{c}_B &= f_B(c) - \int_\Gamma \gamma_B(c_B - h_B(u_B))\, d\sigma \\
\dot{c}_C &= f_C(c) - kc_C c_D - k\lambda c_C c_E \\
\dot{c}_D &= f_D(c) - kc_C c_D \\
\dot{c}_E &= f_E(c) - k\lambda c_C c_E
\end{aligned}$$

with $c = (c_A, \ldots, c_E)$; for the meaning of the different terms as well as reasonable assumptions see §6.3 and Example 7.1.

The system above is obtained under the simplifying assumption that the effect of diffusion of the species C, D and E into the pellets is negligible compared to the influence of the fast reaction of these species inside the bulk phase. This assumption is reasonable in practice, in fact it may happen that some of the species cannot diffuse into the pellets due to the specific pore structure. By the considerations given in Example 7.1 it is then plausible that in the limiting case $k \to \infty$, this process is described by the following system.

$$\begin{aligned}
\partial_t u_A &= \Delta\varphi_A(u_A) - r(u_A,u_B) \text{ in } \Omega, \quad \partial_\nu \varphi_A(u_A) = \gamma_A(c_A - h_A(u_A)) \text{ on } \Gamma \\
\partial_t u_B &= \Delta\varphi_B(u_B) - r(u_A,u_B) \text{ in } \Omega, \quad \partial_\nu \varphi_B(u_B) = \gamma_B(c_B - h_B(u_B)) \text{ on } \Gamma \\
\dot{c}_A &= f_A(c) + f_D(c) - g_D(c) - \int_\Gamma \gamma_A(c_A - h_A(u_A))\, d\sigma \\
\dot{c}_B &= f_B(c) - \int_\Gamma \gamma_B(c_B - h_B(u_B))\, d\sigma \\
\dot{c}_C &= g_C(c) \\
\dot{c}_D &= g_D(c) \\
\dot{c}_E &= g_E(c)
\end{aligned}$$

with discontinuous $g = (g_C, g_D, g_E)$ given by

$$g(c) = \begin{cases} \Big(f_C(c) - f_D(c) - f_E(c), 0, 0\Big) & \text{if } c_C > 0, c_D = c_E = 0 \\ \Big(0, f_D(c) - \dfrac{c_D}{c_D + \lambda c_E} f_C(c), f_E(c) - \dfrac{\lambda c_E}{c_D + \lambda c_E} f_C(c)\Big) & \text{if } c_C = 0, c_D + c_E > 0. \end{cases}$$

If g is replaced by its usc regularization G (corresponding to F in (5)), the resulting version of the limiting problem admits an abstract formulation of the type

$$u' \in -Au + F(u) \quad \text{on } \mathbb{R}_+, \quad u(0) = u_0^\infty$$

in $X = L^1(\Omega)^2 \times \mathbb{R}^5$ with $u = (u_A, u_B, c)$, where A is m-accretive and F is ϵ-δ-usc on every subset $K \times \mathbb{R}^5$ of X with K bounded in $L^\infty(\Omega)^2$; these facts follow from the results in §6.

So far we have been able to carry out the limiting process in the following special case: the semigroup generated by $-A$ is compact (which can be shown if φ_A, φ_B are continuously differentiable on $\mathbb{R} \setminus \{0\}$ such that φ_A', φ_B' satisfy the estimate mentioned behind Lemma 6.1), the initial value u_0 belongs to the "limiting manifold" (i.e. $c_C^0 c_D^0 = c_C^0 c_E^0 = 0$, and then $u_0^\infty = u_0$), the system is decoupled (in the sense that f_C, f_D, f_E are independent of c_A, c_B) and $f_C(0) \neq f_D(0) + f_E(0)$. In this particular situation convergence of $z^k = (c_C^k, c_D^k, c_E^k)$ to the solution of $\dot z \in G(z)$ is a consequence of Example 7.1, and $z(\cdot)$ cannot come to rest at zero due to the last assumption above. Then it is not difficult to show that $\phi_k = k c_C^k c_D^k$ satisfies $\phi_k \rightharpoonup \phi$ in $L^1([0,T])$ for every $T > 0$, and $\phi(t) = f_D(z(t)) - g_D(z(t))$ a.e. on $[0,T]$; notice especially that

$$\frac{d}{dt}(c_C^k c_D^k) \le f_C(z^k) c_D^k + f_D(z^k) c_C^k - k(c_C^k + c_D^k + \lambda c_E^k) c_C^k c_D^k,$$

and $z^k(\cdot)$ is bounded on $[0,T]$ uniformly with respect to $k > 0$ since we obtain a priori estimates on $|u_A|_1 + |u_B|_1 + |c|_1$ due to mass conservation. As a consequence of the compact semigroup, the sequence of solutions $\big((u_A^k, u_B^k, c_A^k, c_B^k)\big)$ of the first part of the system is relatively compact in $C([0,T]; L^1(\Omega)^2 \times \mathbb{R}^2)$, and every accumulation point is a weak solution of the limiting problem. Since uniqueness of weak solutions of the quasi-autonomous problems associated with the particular m-accretive operator given by (18) in §6.3 can be obtained by means of the techniques from Brezis/Crandall [30], it follows that the whole sequence converges to the unique solution of the limiting problem.

Let us finally mention that other related models lead (at least heuristically) to limiting problems in which the pde-part corresponds to an accretive operator in an appropriate L^1-setting that is not m-accretive but satisfies the range condition; see also Remark 7.2 below in this respect.

7.2 Reactions of diffusive species

In the present section we consider a single irreversible reaction $A + B \to P$ between mobile species that takes place inside an isolated vessel (or pellet), represented by a certain bounded region Ω with smooth boundary Γ. This leads to the following model problem, where u_1 and

u_2 denote the concentrations of A and B, respectively.

$$
\begin{aligned}
&\frac{\partial u_1}{\partial t} = \Delta\varphi_1(u_1) - k\, r(u_1,u_2) && \text{in } (0,\infty)\times\Omega \\
&\frac{\partial u_2}{\partial t} = \Delta\varphi_2(u_2) - k\, r(u_1,u_2) && \text{in } (0,\infty)\times\Omega \\
&\frac{\partial\varphi_1(u_1)}{\partial\nu} = \frac{\partial\varphi_2(u_2)}{\partial\nu} = 0 && \text{on } (0,\infty)\times\Gamma \\
&u_1(0,\cdot) = u_{0,1},\ u_2(0,\cdot) = u_{0,2} && \text{in } \Omega
\end{aligned}
\tag{9}
$$

with continuous, strictly increasing $\varphi_k : \mathbb{R}\to\mathbb{R}$ such that $\varphi_k(0)=0$, rate constant $k>0$ and a continuous rate function $r:\mathbb{R}_+^2\to\mathbb{R}$ such that

$$
r(\cdot,\cdot) \text{ is increasing in both variables with } r(a,b)=0 \text{ iff } ab=0; \tag{10}
$$

recall that the latter is a realistic assumption for rate functions.

As before, we are interested in the singular limit $k\to\infty$, corresponding to the case of an instantaneous reaction. We consider (9) as the abstract evolution equation

$$
u' + Au = F_k(u) \ \text{ on } \mathbb{R}_+, \quad u(0)=u_0 \tag{11}
$$

in $X = L^1(\Omega)^2$, and use nonlinear semigroup theory to carry out the limiting process; here A and F_k are defined by

$$
A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\Delta\varphi_1(u_1) \\ -\Delta\varphi_2(u_2) \end{pmatrix} \quad \text{with}
$$

$$
D(A) = \{u\in X : \varphi_i(u_i)\in W^{1,1}(\Omega), \Delta\varphi_i(u_i)\in L^1(\Omega), \frac{\partial\varphi_i(u_i)}{\partial\nu} = 0 \text{ on } \Gamma \text{ for } i=1,2\},
$$

$$
F_k\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(x) = \begin{pmatrix} -k\, r(u_1(x),u_2(x)) \\ -k\, r(u_1(x),u_2(x)) \end{pmatrix} \quad \text{on } D(F_k) = \{u\in X : r(u_1(\cdot),u_2(\cdot))\in L^1(\Omega)\};
$$

cf. §6.1 concerning the precise definition of A. Given an initial value $u_0\in L^\infty(\Omega;\mathbb{R}_+^2)$ we prove that the corresponding mild solutions $u^k(\cdot)$ of (11) satisfy $u_1^k(t)\to w^+(t)$ and $u_2^k(t)\to w^-(t)$ in $L^1(\Omega)$ uniformly on compact subsets of $(0,\infty)$. Here w^+, w^- denotes the positive, respectively negative part of the mild solution $w(\cdot)$ of

$$
\frac{\partial w}{\partial t} = \Delta\varphi(w) \ \text{ in } (0,\infty)\times\Omega, \quad \frac{\partial\varphi(w)}{\partial\nu} = 0 \ \text{ on } (0,\infty)\times\Gamma, \ \ w(0,\cdot)=w_0 \ \text{ in } \Omega \tag{12}
$$

with $w_0 = u_{0,1} - u_{0,2}$ and $\varphi:\mathbb{R}\to\mathbb{R}$ given by

$$
\varphi(r) = \begin{cases} \varphi_1(r) & \text{if } r\ge 0 \\ -\varphi_2(-r) & \text{if } r<0. \end{cases} \tag{13}
$$

Theorem 7.1 *Let $\Omega \subset \mathbb{R}^n$ be open bounded with C^2-boundary Γ. Let $X = L^1(\Omega)^2$ with $|u| = |u_1|_1 + |u_2|_1$ and $A : D(A) \subset X \to X$ as well as $F_k : D(F_k) \to X$ be defined as above, where $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ are continuous, strictly increasing with $\varphi_k(0) = 0$, and $r : \mathbb{R}^2_+ \to \mathbb{R}$ is continuous satisfying (10). Given $u_0 \in L^\infty(\Omega; \mathbb{R}^2_+)$ the initial value problem (11) has a unique mild solution $u^k(\cdot)$ for every $k > 0$, and*

$$\begin{pmatrix} u_1^k(t) \\ u_2^k(t) \end{pmatrix} \to \begin{pmatrix} w^+(t) \\ w^-(t) \end{pmatrix} \quad \text{in } C([\delta,\tau];X) \text{ as } k \to \infty \text{ for all } 0 < \delta < \tau,$$

where $w(\cdot)$ is the mild solution of (12) with φ from (13) and $w_0 = u_{0,1} - u_{0,2}$. If the initial value u_0 satisfies $u_{0,1}u_{0,2} = 0$ then $\delta = 0$ is admissible.

Proof. 1. Given $u_0 \in L^\infty(\Omega; \mathbb{R}^2_+)$ and $k > 0$, existence of a mild solution of (11) can be obtained by means of Theorem 4.2(b) as follows. First of all, it suffices to consider initial value problem (11) on $J = [0, T]$, with arbitrary $T > 0$, instead of $\mathbb{R}_+$. Let

$$K = \{u \in X : 0 \le u_1(x) \le |u_{0,1}|_\infty,\ 0 \le u_2(x) \le |u_{0,2}|_\infty \text{ a.e. in } \Omega\}.$$

Evidently K is closed with $K \subset D(F_k)$, F_k is continuous and bounded on K, and $F_k(u) \in T_K(u)$ on K as a direct consequence of (10). Furthermore, $F_k(K)$ is a bounded subset of $L^\infty(\Omega)^2$, hence $F_k(K)$ is weakly relatively compact in X. Due to Lemma 6.1, the operator A is m-accretive in X and satisfies $(I + \lambda A)^{-1}K \subset K$ for all $\lambda > 0$. Let $\mathcal{S} : L^1(J; X) \to C(J; X)$ denote the solution operator of the quasi-autonomous problem associated with A. Then $\mathcal{S}$ maps weakly relatively compact sets into relatively compact sets, which follows from Lemma 6.2(a) and the remark given behind this lemma. Consequently, Theorem 4.2 applies and yields a mild solution of (11) on J.

To obtain unique solvability of (11) notice that the bracket in X is given by

$$[u, v] = \max\Big(\int_\Omega v_1 \operatorname{Sgn} u_1\, dx + \int_\Omega v_2 \operatorname{Sgn} u_2\, dx\Big) \quad \text{for } u, v \in X$$

where, e.g., $\operatorname{Sgn} u_1$ is short for $\{w \in L^1(\Omega) : w(x) \in \operatorname{Sgn}(u_1(x)) \text{ a.e. on } \Omega\}$. Hence

$$[u - \overline{u}, F_k(u) - F_k(\overline{u})] = \max\Big(-k\int_\Omega (r(u_1, u_2) - r(\overline{u}_1, \overline{u}_2))(\operatorname{Sgn}(u_1 - \overline{u}_1) + \operatorname{Sgn}(u_2 - \overline{u}_2))dx\Big)$$

$$= -k \min\Big(\int_\Omega (r(u_1, u_2) - r(\overline{u}_1, \overline{u}_2))(\operatorname{Sgn}(u_1 - \overline{u}_1) + \operatorname{Sgn}(u_2 - \overline{u}_2))dx\Big) \le 0;$$

recall that $r(a, b)$ is increasing in a, b and observe that $(r(a, b) - r(\overline{a}, b))(\operatorname{sgn}(a - \overline{a}) + \beta) \ge 0$ for $a \ne \overline{a}$ and every $\beta \in [-1, 1]$. Therefore $-F_k$ is s-accretive, hence (11) has a unique mild solution $u(\cdot)$ and $u(t) \in K$ on $\mathbb{R}_+$.

2. Let $\hat{r} : \mathbb{R}^2 \to \mathbb{R}$ be given by $\hat{r}(a, b) = r(p(a), q(b))$ where $p, q : \mathbb{R} \to \mathbb{R}$ denote the metric projections on $[0, |u_{0,1}|_\infty]$, respectively $[0, |u_{0,2}|_\infty]$. Evidently $\hat{r}$ is continuous, bounded

and increasing in both variables. By the previous step the solution of (11) remains the same if we replace $-F_k$ by

$$B_k : X \to X \quad \text{with } B_k \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (x) = \begin{pmatrix} k\,\hat{r}(u_1(x), u_2(x)) \\ k\,\hat{r}(u_1(x), u_2(x)) \end{pmatrix},$$

and B_k is continuous, everywhere defined and also accretive. Therefore $A_k := A + B_k$ with $D(A_k) = D(A)$ defines an m-accretive operator in X for every $k > 0$ by Theorem 5.5. Hence $-A_k$ generates a nonlinear semigroup $T_k(t)$ and the mild solution of (11) is given by $u^k(t) = T_k(t)u_0$ on $\mathbb{R}_+$.
We are going to apply Theorem 1.4 where the first step is to show that

$$J_\lambda \overline{u} := \lim_{k\to\infty} (I + \lambda A_k)^{-1}\overline{u} \ \text{ exists for all } \overline{u} \in L^\infty(\Omega; \mathbb{R}^2_+). \tag{14}$$

For this purpose let $\overline{u} \in L^\infty(\Omega; \mathbb{R}^2_+)$, $\lambda > 0$ and $k > 0$ be given and consider the resolvent equations

$$u_1^k - \lambda\Delta\varphi_1(u_1^k) + \lambda k\,\hat{r}(u_1^k, u_2^k) = \overline{u}_1 \quad \text{in } \Omega, \quad \frac{\partial\varphi_1(u_1^k)}{\partial\nu} = 0 \quad \text{on } \Gamma$$
$$u_2^k - \lambda\Delta\varphi_2(u_2^k) + \lambda k\,\hat{r}(u_1^k, u_2^k) = \overline{u}_2 \quad \text{in } \Omega, \quad \frac{\partial\varphi_2(u_2^k)}{\partial\nu} = 0 \quad \text{on } \Gamma.$$

Let $A_0 v$ denote the operator in $L^1(\Omega)$ corresponding to $-\Delta\varphi_1(v)$ with homogeneous Neumann boundary condition; cf. §6.1 for the precise definition. Then

$$u_1^k = (I + \lambda A_0)^{-1}(\overline{u}_1 - \lambda k\hat{r}(u_1^k, u_2^k))$$

yields $u_1^k \le |\overline{u}_1|_\infty$ by Lemma 6.1 since $\hat{r}(\cdot,\cdot) \ge 0$. To obtain $u_1^k \ge 0$ let $R : L^1(\Omega) \to L^1(\Omega)$ be given by $(Rv)(x) = k\hat{r}(v(x), u_2^k(x))$ where $k > 0$ is fixed. Then $A_0 + R$ is T-accretive in $L^1(\Omega)$ since A_0 is T-accretive and R satisfies

$$\min\Big(\int_\Omega (Rv - R\overline{v})H(v - \overline{v})dx\Big) \ge 0 \quad \text{for } v, \overline{v} \in L^1(\Omega).$$

Consequently, the resolvents of $A_0 + R$ are order-preserving and therefore

$$u_1^k = (I + \lambda(A_0 + R))^{-1}\overline{u}_1 \ge (I + \lambda(A_0 + R))^{-1}(0) = 0.$$

Integration of the first resolvent equation over Ω yields

$$|u_1^k|_1 + \lambda k|\hat{r}(u_1^k, u_2^k)|_1 = |\overline{u}_1|_1,$$

and multiplication of the same equation by $\varphi_1(u_1^k)$ and integration over Ω implies

$$\lambda|\nabla\varphi_1(u_1^k)|_2^2 \le |\overline{u}_1\,\varphi(u_1^k)|_1 \le |\overline{u}_1|_1|\varphi_1(u_1^k)|_\infty \le |\overline{u}_1|_1\varphi_1(|\overline{u}_1|_\infty).$$

Together with the analogous inequalities for u_2^k it follows that $(\varphi_1(u_1^k))_{k>0}$, $(\varphi_2(u_2^k))_{k>0}$ are bounded in $W^{1,2}(\Omega)$, hence relatively compact in $L^2(\Omega)$. Due to the L^∞-bounds for u^k this also yields relative compactness of (u^k) in $L^2(\Omega)$. Therefore, given any sequence $k_j \to \infty$, there is a subsequence of (u^{k_j}) which is denoted by (u^{k_l}) for simplicity, such that

$$u_1^{k_l} \to u_1, \;\; u_2^{k_l} \to u_2, \;\; \nabla\varphi_1(u_1^{k_l}) \rightharpoonup \nabla\varphi_1(u_1), \;\; \nabla\varphi_2(u_2^{k_l}) \rightharpoonup \nabla\varphi_2(u_2) \;\text{ in } L^2(\Omega).$$

It will be shown below that the limit u is uniquely determined, which implies that the original sequence (u^{k_j}) converges to u for arbitrary $k_j \to \infty$, hence (14) holds. Now observe that $u_1, u_2 \geq 0$ a.e. in Ω and also $u_1 u_2 = 0$ a.e. in Ω; the latter follows from $\hat{r}(u_1^k, u_2^k) \to \hat{r}(u_1, u_2)$ in $L^1(\Omega)$ and $|\hat{r}(u_1^k, u_2^k)|_1 \to 0$ which yields $\hat{r}(u_1, u_2) = 0$ a.e. in Ω. Hence u_1 and u_2 are given as $u_1 = w^+$ and $u_2 = w^-$, respectively, if we let $w = u_1 - u_2$. It is consequently sufficient to show that w is uniquely determined. For this purpose let $w^l = u_1^{k_l} - u_2^{k_l}$ and $\overline{w} = \overline{u}_1 - \overline{u}_2$. Multiplication of the resolvent equations by $\phi \in C^1(\overline{\Omega})$ and integration over Ω yields

$$\int_\Omega w^l \phi \, dx + \lambda \int_\Omega \langle \nabla\varphi_1(u_1^{k_l}) - \nabla\varphi_2(u_2^{k_l}), \nabla\phi\rangle dx = \int_\Omega \overline{w}\phi \, dx \;\text{ for all } l \geq 1,$$

hence $l \to \infty$ implies

$$\int_\Omega w\phi \, dx + \lambda \int_\Omega \langle \nabla\varphi_1(u_1) - \nabla\varphi_2(u_2), \nabla\phi\rangle dx = \int_\Omega \overline{w}\phi \, dx \;\text{ for all } \phi \in C^1(\overline{\Omega}).$$

Now recall (see e.g. Chapter II.4 in Ladyzenskaja et al. [75]) that

$$\nabla h^+ = \nabla h \;\text{ a.e. in } \{h > 0\}, \;\; \nabla h^- = -\nabla h \;\text{ a.e. in } \{h < 0\}, \;\; \nabla h = 0 \;\text{ a.e. in } \{h = 0\}$$

for $h \in W^{1,1}(\Omega)$. Due to $u_1 u_2 = 0$ a.e. in Ω we therefore obtain

$$\begin{aligned}
&\nabla(\varphi_1(u_1) - \varphi_2(u_2)) = \nabla\varphi_1(u_1) = \nabla\varphi_1(w) = \nabla\varphi(w) && \text{a.e. in } \{u_1 > 0\} = \{w > 0\},\\
&\nabla(\varphi_1(u_1) - \varphi_2(u_2)) = -\nabla\varphi_2(u_2) = -\nabla\varphi_2(-w) = \nabla\varphi(w) && \text{a.e. in } \{u_2 > 0\} = \{w < 0\},\\
&\nabla(\varphi_1(u_1) - \varphi_2(u_2)) = 0 = \nabla\varphi(w) && \text{a.e. in } \{u_1 = u_2 = 0\} = \{w = 0\},
\end{aligned}$$

where φ is given by (13). Therefore w satisfies

$$\int_\Omega w\phi \, dx + \lambda \int_\Omega \langle \nabla\varphi(w), \nabla\phi\rangle dx = \int_\Omega \overline{w}\phi \, dx \;\text{ for all } \phi \in C^1(\overline{\Omega}),$$

i.e. w is a solution of the resolvent equation $w + \lambda Bw = \overline{w}$, where the operator B is defined by

$$\begin{aligned}
&Bw = -\Delta\varphi(w) \quad \text{on}\\
&D(B) = \{w \in L^1(\Omega) : \varphi(w) \in W^{1,1}(\Omega), \Delta\varphi(w) \in L^1(\Omega), \frac{\partial\varphi(w)}{\partial\nu} = 0 \;\text{ on } \Gamma\};
\end{aligned}$$

cf. definition (2) in §6.1. Consequently w is uniquely determined as $w = (I + \lambda B)^{-1}\overline{w}$ since B is m-accretive by Lemma 6.1, and therefore (14) is valid. Actually we obtained somewhat more, namely

$$\left.\begin{aligned}&J_\lambda\overline{u} := \lim_{k\to\infty}(I+\lambda A_k)^{-1}\overline{u} \ \text{ exists for all } \overline{u}\in X_+ := L^1(\Omega;\mathbb{R}^2_+),\\ &J_\lambda\overline{u}\in D := \{u\in L^1(\Omega)^2 : u_1,u_2\geq 0, u_1u_2=0\} \ \text{ on } X_+,\\ &LJ_\lambda = (I+\lambda B)^{-1}L \ \text{ on } X_+ \text{ with } Lu := u_1-u_2;\end{aligned}\right\} \tag{15}$$

notice that convergence on X_+ follows from (14) since the resolvents of A_k are nonexpansive and $L^\infty(\Omega;\mathbb{R}^2_+)$ is dense in X_+.

3. By means of (15) we are able to obtain convergence of $(u^k(t))$ in case the initial value u_0 satisfies $u_{0,1}u_{0,2}=0$. Define the operator A_∞ in X by means of

$$\mathrm{gr}\,(A_\infty) = \{\Big(J_\lambda\overline{u}, \frac{1}{\lambda}(\overline{u}-J_\lambda\overline{u})\Big) : \lambda>0, \overline{u}\in X_+\}.$$

From the first line in (15) it follows immediately that $A_\infty \subset \liminf_{k\to\infty} A_k$, hence A_∞ is in particular accretive. By definition of A_∞ it is also clear that J_λ are the resolvents of A_∞ and $R(I+\lambda A_\infty)\supset X_+$ for all $\lambda>0$. Moreover, the second line in (15) yields $D(A_\infty)\subset D$ and then $\overline{D(A_\infty)} = D$ follows from

$$\overline{D(A_\infty)} = \overline{\{(w^+,w^-) : w\in D(B)\}} \supset \{(w^+,w^-) : w\in L^1(\Omega)\} = D.$$

Therefore A_∞ is accretive and satisfies the range condition, hence $-A_\infty$ generates a semigroup $T(t)$ of nonexpansive mappings on D. Then applications of Theorem 1.4 shows that

$$(u_0^k)\subset X_+ \text{ with } u_0^k\to u_0\in D \ \text{ implies } \ T_k(t)u_0^k \to T(t)u_0 \ \text{ on } \mathbb{R}_+, \tag{16}$$

where the convergence is uniform on bounded sets.

It remains to identify the semigroup $T(t)$. For this purpose notice that $L_{|D} : D\to L^1(\Omega)$ is invertible with $L_{|D}^{-1}(w) = (w^+,w^-)$. Hence $(I+\lambda A_\infty)^{-1} = L_{|D}^{-1}(I+\lambda B)^{-1}L$ by (15) yields

$$L(I+\lambda A_\infty)^{-n} = (I+\lambda B)^{-n}L \ \text{ on } X_+ \text{ for all } n\geq 1.$$

Let $S(t)$ denote the semigroup generated by $-B$ on $L^1(\Omega)$. Since $S(t)$ and $T(t)$ are given by the exponential formula, the equation above implies

$$LT(t) = S(t)L \ \text{ on } D \text{ for } t\geq 0,$$

hence

$$T(t)\begin{pmatrix}u_{0,1}\\ u_{0,2}\end{pmatrix} = \begin{pmatrix}w^+(t)\\ w^-(t)\end{pmatrix} \ \text{ with } w(t) = S(t)(u_{0,1}-u_{0,2}).$$

Therefore

$$\begin{pmatrix}u_1^k(t)\\ u_2^k(t)\end{pmatrix} \to \begin{pmatrix}w^+(t)\\ w^-(t)\end{pmatrix} \ \text{ in } C([0,\tau];X) \text{ as } k\to\infty \text{ for all } \tau>0$$

if the initial value belongs to D, i.e. in case $u_0 \in L^\infty(\Omega;\mathbb{R}^2_+)$ with $u_{0,1}u_{0,2} = 0$.

4. Given an arbitrary initial value $u_0 \in L^\infty(\Omega;\mathbb{R}^2_+)$, let $w_0 = u_{0,1} - u_{0,2}$ and $0 < \delta < \tau$. We then have to show

$$\begin{pmatrix} u_1^k(t) \\ u_2^k(t) \end{pmatrix} = T_k(t)\begin{pmatrix} u_{0,1} \\ u_{0,2} \end{pmatrix} \to \begin{pmatrix} w^+(t) \\ w^-(t) \end{pmatrix} = T(t)\begin{pmatrix} w_0^+ \\ w_0^- \end{pmatrix} \quad \text{uniformly on } [\delta,\tau] \text{ as } k \to \infty,$$

and the latter follows from (16) if

$$T_k(\delta)\begin{pmatrix} u_{0,1} \\ u_{0,2} \end{pmatrix} \to T(\delta)\begin{pmatrix} w_0^+ \\ w_0^- \end{pmatrix} \quad \text{as } k \to \infty. \tag{17}$$

To obtain (17) consider $\hat{u}^k(t) = u^k(t/k)$. Exploiting the fact that $u^k(\cdot)$ is the integral solution of

$$u' + \Big[A + k\, B_1\Big]u = 0 \ \text{ on } \mathbb{R}_+, \ \ u(0) = u_0$$

it follows immediately that $\hat{u}^k(\cdot)$ is the integral solution of

$$\hat{u}' + \Big[\frac{1}{k}A + B_1\Big]u = 0 \ \text{ on } \mathbb{R}_+, \ \ \hat{u}(0) = u_0.$$

To obtain convergence of $(\hat{u}^k)$, we compute $L := \liminf_{k\to\infty}\Big[\frac{1}{k}A + B_1\Big]$. Evidently $B_1 u \in Lu$ for all $u \in D(A)$, hence $B_1 u \in Lu$ on $\overline{D(A)} = X$ since $B : X \to X$ is continuous and L has closed graph. Therefore $\mathrm{gr}\,(B_1) \subset \mathrm{gr}\,(L)$ and then $B_1 = L$ since B_1 is m-accretive, hence especially maximal accretive, and L is accretive. Consequently, Theorem 1.4 yields

$$\hat{u}^k \to \hat{u} \ \text{ in } C([0,\sigma];X) \text{ as } k \to \infty \text{ for all } \sigma > 0,$$

where $\hat{u}(\cdot)$ is the solution of

$$\hat{u}' + B_1\hat{u} = 0 \ \text{ on } \mathbb{R}_+, \ \ \hat{u}(0) = u_0. \tag{18}$$

For the subsequent argumentation we need to know the asymptotic behavior of $\hat{u}(t)$, and here it is helpful to observe that (18) is in fact a family of ordinary differential equations, parameterized by $x \in \Omega$. We therefore consider

$$\begin{aligned} a' &= -\hat{r}(a,b) \ \text{ on } \mathbb{R}_+, \quad a(0) = a_0 \geq 0 \\ b' &= -\hat{r}(a,b) \ \text{ on } \mathbb{R}_+, \quad b(0) = b_0 \geq 0. \end{aligned} \tag{19}$$

Now recall that $\hat{r} : \mathbb{R}^2 \to \mathbb{R}$ is continuous and increasing in both variables with $\hat{r}(a,b) = 0$ if $a \leq 0$ or $b \leq 0$, and $\hat{r}(a,b) = 0$ with $a,b \geq 0$ implies $ab = 0$. In particular, the right-hand side in (19) is dissipative in $(\mathbb{R}^2, |\cdot|_1)$. Hence (19) has a unique solution $(a(t),b(t)) = (a(t;a_0,b_0), b(t;a_0,b_0))$ for every a_0, b_0. Evidently $a(t) \searrow a_\infty \geq 0$ and $b(t) \searrow b_\infty \geq 0$ with $\hat{r}(a_\infty,b_\infty) = 0$, i.e. $a_\infty b_\infty = 0$. Furthermore $a_\infty - b_\infty = a_0 - b_0$ since $(a-b)' = 0$ on $\mathbb{R}_+$, hence $a_\infty = (a_0 - b_0)^+$ and $b_\infty = (a_0 - b_0)^-$. Consequently,

$$a(t;a_0,b_0) \to (a_0 - b_0)^+, \ \ b(t;a_0,b_0) \to (a_0 - b_0)^- \quad \text{as } t \to \infty. \tag{20}$$

Let $\hat{u}_1(t)(x) = a(t; u_{0,1}(x), u_{0,2}(x))$ and $\hat{u}_2(t)(x) = b(t; u_{0,1}(x), u_{0,2}(x))$ for $t \geq 0$ and $x \in \Omega$. Then $\hat{u}$ is the strong solution of (18), hence (20) together with the dominated convergence theorem implies

$$\begin{pmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{pmatrix} \to \begin{pmatrix} (u_{0,1} - u_{0,2})^+ \\ (u_{0,1} - u_{0,2})^- \end{pmatrix} = \begin{pmatrix} w_0^+ \\ w_0^- \end{pmatrix} \quad \text{in } X \text{ as } t \to \infty.$$

Now we are able to obtain (17) as follows. Given $\epsilon > 0$, there is $\sigma > 0$ such that

$$\left| \begin{pmatrix} \hat{u}_1(\sigma) \\ \hat{u}_2(\sigma) \end{pmatrix} - \begin{pmatrix} w_0^+ \\ w_0^- \end{pmatrix} \right| \leq \epsilon, \quad \text{hence} \left| \begin{pmatrix} \hat{u}_1^k(\sigma) \\ \hat{u}_2^k(\sigma) \end{pmatrix} - \begin{pmatrix} w_0^+ \\ w_0^- \end{pmatrix} \right| \leq 2\epsilon \;\text{ for all } k \geq k_\epsilon.$$

Translated to the original time scale the latter means

$$\left| T_k(\sigma/k) \begin{pmatrix} u_{0,1} \\ u_{0,2} \end{pmatrix} - \begin{pmatrix} w_0^+ \\ w_0^- \end{pmatrix} \right| \leq 2\epsilon \;\text{ for all } k \geq k_\epsilon,$$

which implies

$$\left| T_k(\delta) \begin{pmatrix} u_{0,1} \\ u_{0,2} \end{pmatrix} - T_k(\delta - \sigma/k) \begin{pmatrix} w_0^+ \\ w_0^- \end{pmatrix} \right| \leq 2\epsilon \;\text{ for all large } k.$$

Since (w_0^+, w_0^-) belongs to D, exploitation of (16) yields

$$T_k(t) \begin{pmatrix} w_0^+ \\ w_0^- \end{pmatrix} \to T(t) \begin{pmatrix} w_0^+ \\ w_0^- \end{pmatrix} \quad \text{uniformly on } [0, \delta],$$

and therefore

$$\left| T_k(\delta) \begin{pmatrix} u_{0,1} \\ u_{0,2} \end{pmatrix} - T(\delta) \begin{pmatrix} w_0^+ \\ w_0^- \end{pmatrix} \right| \leq 3\epsilon \;\text{ for all large } k.$$

Consequently (17) holds which ends the proof. □

7.3 Remarks

Remark 7.1 Theorem 7.1 is Theorem 1 in Bothe [26] and contains the main result in Evans [51] where the following special case of system (9) has been considered.

$$\begin{aligned} & u_t = d_1 \Delta u - kuv && \text{in } (0,\infty) \times \Omega \\ & v_t = d_2 \Delta v - kuv && \text{in } (0,\infty) \times \Omega \\ & \partial_\nu u = \partial_\nu v = 0 && \text{on } (0,\infty) \times \partial\Omega \\ & u(0,\cdot) = u_0, \; v(0,\cdot) = v_0 && \text{in } \Omega. \end{aligned}$$

In this paper convergence of (u^k, v^k) to (w^+, w^-) in $L^1((0,T) \times \Omega)$ is obtained in the regular case when the initial values satisfy $u_0 v_0 = 0$. Observe that we cannot expect to obtain convergence of u^k to a limit u^∞ in $C([0,\tau]; X)$ in the situation of Theorem 7.1 with arbitrary

$u_0 \in L^\infty(\Omega;\mathbb{R}^2_+)$, since a jump at $t = 0$ develops as $k \to \infty$; notice that $u^\infty(0) = u_0$ but $u^\infty(0+) = ((u_{0,1} - u_{0,2})^+, (u_{0,1} - u_{0,2})^-)$. This phenomenon is intuitively clear from the physical background: in the limiting case $k = \infty$ the concentrations of A and B will be instantaneously reduced at every point by such an amount that one of them vanishes. Thereby a separating interface develops which then starts to move, driven by diffusion of A and B towards this free boundary.

Remark 7.2 In Hilhorst/v.d. Hout/Peletier [67] the authors study a problem that is related to Theorem 7.1, namely the instantaneous limit for a single irreversible reaction between a mobile and an immobile species. Further assumptions lead to the following reaction-diffusion system with one spatial dimension.

$$\begin{aligned} &u_t = u_{xx} - kuv, \quad v_t = -kuv \quad \text{for } t > 0, x > 0 \\ &u(t,0) = \psi(t), \quad u(0,\cdot) = 0, \quad v(0,\cdot) \equiv v_0 > 0. \end{aligned}$$

Under appropriate conditions on ψ, convergence of u^k, v^k to certain limit concentrations u, v is obtained. Furthermore it is shown that a free boundary (given by a single point $\rho(t)$ due to the one-dimensional setting) develops as $k \to \infty$, which separates the two regions where $u > 0$ and $v = 0$, respectively $u = 0$ and $v = v_0$. Here the limit problem for u, ρ turns out to be a classical one phase Stefan problem.

Let us note that several processes in Chemical Engineering lead to related moving boundary problems, called core-shell models in this context. For a realistic model of such a process additional aspects like macroscopic convection and mass transfer resistance have to be taken into account. The following model occurs for example in semibatch regeneration of exhausted ion exchangers and has been studied in Bothe/Prüss [27].

$$\begin{aligned} &\frac{\partial v}{\partial t} = D\frac{1}{r^2}(r^2 v_r)_r && t > 0, \rho(t) < r < R \\ &\frac{\partial w}{\partial t} = D'\frac{1}{r^2}(r^2 w_r)_r && t > 0, R < r < R+\delta \\ &\frac{d\rho^3}{dt} = -\kappa\rho^2 v_r(t,\rho+) && t > 0 \\ &\frac{dc}{dt} = -\alpha w_r(t, R+\delta) + \beta(\hat{c} - c) && t > 0 \end{aligned} \tag{21}$$

with boundary and initial conditions

$$\begin{gathered} v(t,\rho(t)) = 0,\ v_r(t,0) = 0,\ v(t,R) = w(t,R)/H_A, \\ Dv_r(t,R) = D'w_r(t,R),\ w(t,R+\delta) = c(t), \\ v(0,\cdot) = 0 \text{ in } [0,R],\ w(0,\cdot) = w_0 \text{ in } [R,R+\delta],\ \rho(0) = R,\ c(0) = c_0. \end{gathered} \tag{22}$$

Here v denotes the concentration of the mobile reactand A inside a spherical pellet of radius R, w is the concentration of the same species in a stagnant film of thickness δ around the pellet, ρ denotes the position of the moving boundary that corresponds to the reaction front, c is the bulk concentration and $\hat{c}$ the feed concentration of A; see [27] for further explanations. Here, let us just mention that [27] provides a new approach to this type of "nonstandard" Stefan problems, based on the use of ρ^3 as an explicit system variable which then allows application of the theory of accretive operators and nonlinear semigroups. To be more specific, let

$$X = L^1([0,R], r^2dr) \times L^1([R, R+\delta], r^2dr) \times \mathbb{R}^2,$$

equipped with the norm

$$||u|| = |v|_1 + |w|_1 + \frac{D}{\kappa}|\rho^3| + \frac{D'}{\alpha}(R+\delta)^2|c| \quad \text{for } u = (v, w, \rho^3, c) \in X$$

and the partial ordering

$$u \leq \overline{u} \ \text{ iff } \ v \leq \overline{v} \text{ a.e. in } [0,R], \ w \leq \overline{w} \text{ a.e. in } [R, R+\delta], \ \rho \geq \overline{\rho}, \ c \leq \overline{c}.$$

Define the operator A in X by means of

$$A \begin{pmatrix} v \\ w \\ \rho^3 \\ c \end{pmatrix} = \begin{pmatrix} D\frac{1}{r^2}\partial_r(r^2\partial_r v)\chi_{(\rho,R)} \\ D'\frac{1}{r^2}\partial_r(r^2\partial_r w) \\ -\kappa\rho^2\partial_r v(\rho+) \\ -\alpha\partial_r w(R+\delta) + \beta(\hat{c} - c) \end{pmatrix}$$

with

$$\begin{aligned} D(A) = \{u \in X : \ & v \in W^{1,1}([0,R]), \ v_{|[\rho,R]} \in W^{2,1}([\rho,R]), \ w \in W^{2,1}([R,R+\delta]), \\ & v \geq 0, \ v = 0 \text{ in } (0,\rho), \ v'(0) = 0, \ w \geq 0, \rho \in [0,R], \ c \geq 0, \\ & v(R) = w(R)/H_A, \ Dv'(R) = D'w'(R), \ w(R+\delta) = c\}. \end{aligned}$$

Then A is accretive and T-accretive in X, and satisfies the range condition. Based on these facts, the following result concerning the dynamics of this core-shell process is obtained.

Theorem 7.2 *Suppose that all constants $D, D', H_A, R, \delta, \alpha, \beta, \kappa, \hat{c}$ are strictly positive, and let X, $||\cdot||$, $\leq$ and A be defined as above. Then problem (21), (22) has a unique mild solution $u(t) = S(t)u_0$ for every initial value $u_0 \in \overline{D(A)}$, and the semigroup $S(t)$ is nonexpansive and order-preserving on $\overline{D(A)}$. There is a unique steady state $u_\infty = (\hat{c}/H_A, \hat{c}, 0, \hat{c}) \in D(A)$ which is globally asymptotically stable. Given $u_0 \in \overline{D(A)}$, the free boundary $\rho(t)$ is continuous, decreasing and reaches zero in finite time, i.e. there exists a time $\tau(u_0) \geq 0$ such that $\rho(t) > 0$ is strictly decreasing on $[0, \tau(u_0))$ and $\rho(t) = 0$ on $[\tau(u_0), \infty)$. Furthermore, if $u_0 = (v_0, w_0, \rho_0^3, c_0) \in \overline{D(A)}$ is such that $v_0 \in L^2([0,R], r^2dr)$ and $w_0 \in L^2([R, R+\delta], r^2dr)$, then the corresponding mild solution u is also a strong solution and satisfies*

$$\begin{aligned} & u \in W^{1,\infty}((0,\infty); X), \ v \in C((0,\infty); W^{1,2}(0,R)), \ w \in C((0,\infty); W^{2,2}(R, R+\delta)), \\ & v(t,\cdot) \in W^{2,2}(\rho(t), R) \text{ for each } t > 0, t \neq \tau(u^0), \end{aligned}$$

and the boundary conditions in (22) are valid for all $t > 0$, $t \neq \tau(u^0)$.

Let us finally note that related model problems have been studied in Conrad/Hilhorst/Seidman [37] and Friedman/Ross/Zhang [56].

§8 Instantaneous Reversible Reactions

We continue to study chemically reacting systems involving fast reactions with additional mass transport but, in contrast to the previous paragraph, we consider the case of fast reversible reactions. We are again concerned with the passage to infinite reaction speed, which requires different techniques compared to the irreversible case. For this reason, we start with a compilation of known facts concerning the dynamics of systems of independent reversible reactions (with finite reaction speed) under the assumption of mass-action kinetics. We include a short proof adapted to the special situation considered here, since most of the arguments will also be important in the study of the instantaneous reaction limit.

8.1 Systems of independent reversible reactions

Suppose that m reversible reactions

$$\alpha_{j,1}C_1 + \ldots + \alpha_{j,n}C_n \rightleftharpoons \beta_{j,1}C_1 + \ldots + \beta_{j,n}C_n \quad (j = 1, \ldots, m)$$

take place inside a continuously stirred isolated vessel, involving chemical species $C_1, \ldots, C_n$. Here $\alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,n})$ and $\beta_j = (\beta_{j,1}, \ldots, \beta_{j,n})$ with $\alpha_{j,i}, \beta_{j,i} \in \mathbb{N}_0$ are the *stoichiometric vectors* corresponding to the j^{th} reaction. We always assume independence of the reactions, which means that $\{\nu_1, \ldots, \nu_m\}$ with $\nu_j := \beta_j - \alpha_j$ is a linearly independent subset of $\mathbb{R}^n$. Equivalently, the *stoichiometric matrix* $N = (\nu_1^T, \ldots, \nu_n^T)$ satisfies $\ker(N) = \{0\}$. On the basis of mass-action kinetics the rate function of the j^{th} reaction is given by

$$r_j(c) = k_j(c^{\alpha_j} - \kappa_j c^{\beta_j}) \text{ with } k_j, \kappa_j > 0,$$

where $c^x = \prod_{i=1}^{n} c_i^{x_i}$ for $c \in \mathbb{R}_+^n$ and $x \in \mathbb{N}_0^n$ (with $0^0 := 1$).

In this situation, the time-evolution of the vector $c = (c_1, \ldots, c_n)$ of concentrations of the corresponding species is governed by the initial value problem

$$\dot{c} = \sum_{j=1}^{m} \nu_j r_j(c) \text{ on } \mathbb{R}_+, \quad c(0) = c_0. \tag{1}$$

Since the right-hand side has its values in the *stoichiometric space* $S = \operatorname{Im}(N)$, every solution remains inside the so-called *stoichiometric class* $c_0 + S$ determined by its initial value c_0. Below we will sometimes use the dual formulation of this fact: if E denotes an $(n - m) \times n$-matrix of full rank such that $\ker(E) = S$, then $Ec(t) \equiv Ec_0$ for every solution of (1). In the sequel the letter E will always stand for such a matrix. Evidently $\operatorname{Im}(E^T) = S^\perp$ then, and every $e \in S^\perp \setminus \{0\}$ corresponds to a "conservation law" of the system of chemical reactions. The system is said to be *conservative* if there is $e \in S^\perp$ which is strictly positive (i.e. $e_i > 0$ for all i). The latter holds in practice if all involved species have an atomic structure, and the total number of atoms is conserved under the chemical reactions; see Chapter 3 in Erdi/Toth [49].

In addition to the abbreviation c^x mentioned above, the following notation will be used throughout this section: $c \gg 0$ is short for $c \in \overset{\circ}{\mathbb{R}}{}^n_+$, and

$$g(c) := (g(c_1), \ldots, g(c_n)) \ \text{ for } c \gg 0 \text{ where } g : (0, \infty) \to \mathbb{R},$$

$$h(c, \overline{c}) := (h(c_1, \overline{c}_1), \ldots, h(c_n, \overline{c}_n)) \ \text{ for } c \gg 0 \text{ where } h : (0, \infty)^2 \to \mathbb{R}.$$

So, for example, $e^c = (e^{c_1}, \ldots, e^{c_n})$, $c/\overline{c} = (c_1/\overline{c}_1, \ldots, c_n/\overline{c}_n)$ and $c \cdot \overline{c} = (c_1\overline{c}_1, \ldots, c_n\overline{c}_n)$.

Theorem 8.1 *Let $\alpha_j, \beta_j \in \mathbb{N}_0^n$ for $j = 1, \ldots, m$ be such that $\{\nu_1, \ldots, \nu_m\}$ with $\nu_j = \beta_j - \alpha_j$ is a linearly independent subset of $\mathbb{R}^n$. Let $k_j, \kappa_j > 0$ and $r_j(c) = k_j(c^{\alpha_j} - \kappa_j c^{\beta_j})$ for $j = 1, \ldots, m$. Then the following holds.*

(a) *For every $c_0 \gg 0$ there is a unique equilibrium $\overline{c} \gg 0$ of (1) within the stoichiometric class $c_0 + S$. If $c^* \gg 0$ is any equilibrium, there is $x \in S^\perp$ such that $\overline{c} = c^* \cdot e^x$.*

(b) *For every $c_0 \gg 0$ there is a unique solution of (1). This solution is strictly positive and exists globally.*

(c) *Given $c_0 \gg 0$, the solution $c(\cdot; c_0)$ of (1) satisfies $c(t; c_0) \to \overline{c}$ as $t \to \infty$, where $\overline{c}$ is the equilibrium in the stoichiometric class $c_0 + S$. In addition, there is $\eta = \eta(c_0) > 0$ such that $\min_i c_i(t; c_0) \geq \eta$ on $\mathbb{R}_+$.*

Proof. **(a)** Notice first that $c \gg 0$ is an equilibrium of (1) iff $c^{\nu_j} = 1/\kappa_j$ for all $j = 1, \ldots, m$. Hence $c := e^x$ is a strictly positive equilibrium if x is a solution of $N^T x = y$ with $y = (-\ln \kappa_1, \ldots, -\ln \kappa_m)$, and such a solution exists due to $\operatorname{Im}(N^T) = (\ker(N))^\perp = \{0\}^\perp = \mathbb{R}^n$. Moreover, given an arbitrary stationary solution $c \gg 0$, the set of all equilibria is given as $\{c \cdot e^x : x \in S^\perp\}$. If c, c^* are equilibria in the class $c_0 + S$, then $\ln c - \ln c^* \in S^\perp$ implies

$$c - c^* \perp \ln c - \ln c^* \quad \Leftrightarrow \quad \sum_{i=1}^n (c_i - c_i^*)(\ln c_i - \ln c_i^*) = 0,$$

hence $c = c^*$. Consequently, there is at most one equilibrium in every stoichiometric class.

To prove existence within the class $c_0 + S$, fix any equilibrium $c \gg 0$. It suffices to find $x \in S^\perp$ such that $c \cdot e^x - c_0 \in S$. Define $\phi : \mathbb{R}^n \to \mathbb{R}$ by $\phi(x) = \langle c, e^x \rangle - \langle c_0, x \rangle$. Then $\nabla\phi(x) = c \cdot e^x - c_0$ and $\phi''(x) = \operatorname{diag}(c_i e^{x_i})$ is positive definite, hence ϕ is strictly convex. Consider the set

$$C = \{x \in S^\perp : \phi(x) \leq \phi(0)\}.$$

Evidently C is closed convex with $0 \in C$. To show that C is also bounded, fix $x \in S^\perp$ with $|x| = 1$ and consider $\phi(sx)$ as a function of $s \in \mathbb{R}_+$. Evidently $\phi(sx) = \sum_{i=1}^n \varphi_i(s)$ with $\varphi_i(s) = c_i e^{sx_i} - c_{0,i} s x_i$. Given $a, b > 0$, it is easy to check that $\varphi(r) = ae^r - br$ is strictly convex with $\varphi(r) \geq b(1 - \ln b/a)$ on $\mathbb{R}$. Hence $\phi(sx) \leq \phi(0) = |c|_1$ implies

$$\varphi_i(s) \leq |c|_1 + (n-1)M \quad \text{with } M = \max_{k=1,\ldots,n} c_{0,k} |1 - \ln \frac{c_{0,k}}{c_k}|.$$

Choose $i \in \{1, \dots, n\}$ such that $|x_i| \geq 1/n$. If $x_i > 0$ then

$$\varphi_i(s) \geq c_i(1 + sx_i + \frac{1}{2}s^2x_i^2) - c_{0,i}sx_i \geq c_i - |c_i - c_{0,i}|s + c_i\frac{s^2}{2n^2},$$

hence $sx \in C$ implies

$$c_i - |c_i - c_{0,i}|s + c_i\frac{s^2}{2n^2} \leq |c|_1 + (n-1)M$$

and therefore $s \leq \rho$ for some $\rho > 0$. In case $x_i < 0$ the same conclusion follows from $\varphi_i(s) \geq c_{0,i}s/n$, thus $x \in C$ implies $|x| \leq \rho$ with some $\rho = \rho(c_0, c) > 0$, hence C is compact. Consequently, there is $x \in C$ such that $\phi(x) \leq \phi(z)$ for all $z \in C$. This implies $\langle \nabla\phi(x), \nu\rangle = 0$ for all $\nu \in S^\perp$, and therefore $\nabla\phi(x) = c \cdot e^x - c_0 \in S^{\perp\perp} = S$.

(b) To obtain the second assertion observe that the right-hand side in (1) is only defined on $\mathbb{R}^n_+$ where it is locally Lipschitz continuous. We claim that $g(c) = \sum_{j=1}^m \nu_j r_j(c)$ is quasi-positive. To see this, write $g_i(c)$ as

$$g_i(c) = \sum_{j:\nu_{j,i}<0} \nu_{j,i}r_j(c) + \sum_{j:\nu_{j,i}>0} \nu_{j,i}r_j(c)$$

and notice that for instance $\nu_{j,i} < 0$ and $c_i = 0$ imply $c^{\alpha_j} = 0$ since $\alpha_{j,i} > 0$. Hence

$$\nu_{j,i}r_j(c) = |\nu_{j,i}|k_j\kappa_j c^{\beta_j} \geq 0$$

in this case. Consequently $\mathbb{R}^n_+$ is invariant for $\dot{c} = g(c)$ and therefore (1) has a unique local solution which stays nonnegative; this follows by Corollary 2.1. Let $c(\cdot)$ denote this solution and let $[0, T)$ be its maximal interval of existence. Due to the above arguments concerning the structure of $g_i(c)$ and the fact that $\alpha_{j,i}, \beta_{j,i} \in \mathbb{N}_0$, it is easy to see that

$$\dot{c}_i = -\varphi_i(t)c_i + \psi_i(t) \;\text{ on } [0,T) \quad \text{with continuous } \varphi_i, \psi_i \geq 0.$$

By the variation of constants formula it follows that

$$c_i(t) = c_{0,i}\exp\Big(-\int_0^t \varphi_i(s)ds\Big) + \int_0^t \exp\Big(-\int_s^t \varphi_i(\tau)d\tau\Big)\psi_i(s)ds > 0 \;\text{ on } [0,T).$$

It remains to prove global existence which holds if $c(\cdot)$ is bounded on bounded intervals. For this purpose let $\overline{c} \gg 0$ be the stationary solution of (1) in the class $c_0 + S$ and define $V : (0,\infty)^n \to \mathbb{R}$ by

$$V(c) = \sum_{i=1}^n c_i(\ln c_i - \ln \overline{c}_i) - (c_i - \overline{c}_i). \tag{2}$$

Evidently V is continuously differentiable with $0 \leq V(c) \to \infty$ if $c_i \to \infty$ for some i. We claim that V is a Lyapunov function for (1). Indeed, $\nabla V(c) = \ln(c/\overline{c})$ and therefore

$$\langle \nabla V(c), g(c)\rangle = \sum_{i=1}^n\sum_{j=1}^m \nu_{j,i}\ln\frac{c_i}{\overline{c}_i}\, r_j(c) = \sum_{j=1}^m k_j \ln\frac{c^{\nu_j}}{\overline{c}^{\nu_j}}\,(c^{\alpha_j} - \kappa_j c^{\beta_j}).$$

Due to $\bar{c}^{\nu_j} = 1/\kappa_j$, this implies

$$\langle \nabla V(c), g(c) \rangle = -\sum_{j=1}^{m} k_j c^{\alpha_j} \Big[\Big(\frac{c}{\bar{c}}\Big)^{\nu_j} - 1 \Big] \ln \Big(\frac{c}{\bar{c}}\Big)^{\nu_j} \leq 0. \tag{3}$$

Consequently $V(c(t)) \leq V(c_0)$, hence $c(\cdot)$ is bounded on $[0, T)$. Therefore, the unique solution $c(\cdot)$ exists globally and is bounded on $\mathbb{R}_+$.

(c) By the preceeding step, all semiorbits are relatively compact. Hence $\rho(c(t), M) \to 0$ as $t \to \infty$ by LaSalle's invariance principle (see e.g. Theorem 18.3 in Amann [2]), where $\rho(c, M)$ denotes the distance from c to the set

$$M = \{c^* \in \mathbb{R}_+^n : V(c(\cdot; c^*)) \equiv V(c^*)\}.$$

Due to (3) and the fact that $(x-1)\ln x \geq \dfrac{(x-1)^2}{x+1}$ for all $x > 0$, which follows by the mean value theorem, we obtain

$$\langle \nabla V(c), g(c) \rangle \leq -\sum_{j=1}^{m} k_j \frac{(c^{\alpha_j} - \kappa_j c^{\beta_j})^2}{c^{\alpha_j} + \kappa_j c^{\beta_j}}. \tag{4}$$

Since the solution $c(\cdot)$ of (1) is globally bounded, this yields

$$\langle \nabla V(c), g(c) \rangle \leq -\omega \sum_{j=1}^{m} \frac{1}{k_j} r_j(c)^2 \quad \text{with some } \omega > 0.$$

Thus $c^* \in M$ means $r_j(c^*) = 0$ for all $j = 1, \dots, m$, i.e. M consists of equilibria. On the other hand, the solution $c(\cdot; c_0)$ of (1) with initial value c_0 stays inside $c_0 + S$, since

$$Ec(t) = Ec_0 + \int_0^t Eg(c(s))ds = Ec_0 \quad \text{for all } t \geq 0.$$

Consequently

$$\{c^* : c^* = \lim_{k \to \infty} c(t_k; c_0) \text{ for some } t_k \to \infty\} \subset M \cap (c_0 + S) = \{\bar{c}\},$$

hence $c(t; c_0) \to \bar{c}$ as $t \to \infty$. The other assertion in (c) is obvious, since $c_i(t) \geq \dfrac{1}{2} \min_l \bar{c}_l$ on $[T, \infty)$ with some $T > 0$, and $c(\cdot)$ is continuous on $[0, T]$ with $c(t) \gg 0$ by (b). □

8.2 Reactions with macroscopic convection

In analogy to the situation considered in Example 7.1 we suppose that a system of slow and fast reactions is performed inside a CSTR, but here we assume that the fast reactions are reversible. We are again interested in the limiting case of instantaneous reactions, hence we study the family of initial value problems

$$\dot{c}^k = f(c^k) + kN\Lambda R(c^k) \quad \text{on } \mathbb{R}_+, \;\; c^k(0) = c_0 \tag{5}$$

with a (large) parameter $k \geq 0$, and consider the singular limit as $k \to \infty$. Here $N = (\nu_1^T, \dots, \nu_m^T)$, $\Lambda = \text{diag}(\lambda_1, \dots \lambda_m)$ with $\lambda_j \in (0,1]$ and $R_j(c) = c^{\alpha_j} - \kappa_j c^{\beta_j}$ for $j = 1, \dots, m$. This formulation, i.e. with $k\lambda_j$ instead of k_j, takes care of the fact that the ratios k_j/k_l of the rate constants have to remain fixed as $k_j \to \infty$.

To see what are realistic conditions for f, recall that f splits into $g+h$ where g refers to feeds and h corresponds to additional slow reactions. Here g is typically given by $g(c) = \dfrac{1}{\tau}(c^f - c)$ where $\tau > 0$ is the so-called holding time of the CSTR, and $c^f \in \mathbb{R}^n_+$ denotes the vector of feed concentrations. Assuming mass-action kinetics again, the slow reaction term h is given by

$$h(c) = \sum_{j=m+1}^{N} k_j(\beta_j - \alpha_j)(c^{\alpha_j} - \kappa_j c^{\beta_j}) \quad \text{with } k_j > 0, \kappa_j \geq 0,$$

with $\kappa_j = 0$ if the j^{th} reaction is irreversible. While the concrete structure of g and h is not so important here, it follows that

$$f : \mathbb{R}^n_+ \to \mathbb{R}^n \ \text{ is locally Lipschitz and quasi-positive;} \tag{6}$$

remember the proof of Theorem 8.1(b) and recall that f is called quasi-positive if $c_i = 0$ implies $f_i(c) \geq 0$, i.e. if f satisfies the subtangential condition with respect to $\mathbb{R}^n_+$.
Under the natural assumption that the system of slow reactions is conservative there exists $e \in \overset{\circ}{\mathbb{R}}{}^n_+$ such that $\langle e, \beta_j - \alpha_j \rangle = 0$ for all $j = m+1, \dots, N$. Therefore, since g has linear growth, another reasonable assumption on f is

$$\langle e, f(c) \rangle \leq a(1 + |c|_1) \ \text{ on } \mathbb{R}^n_+ \text{ with } a \geq 0 \text{ and } 0 \ll e \in S^\perp. \tag{7}$$

The following result establishes convergence of the solutions $c^k(\cdot)$ of (5) to the solution $c^\infty(\cdot)$ of the limiting equation

$$\dot{c}^\infty = f(c^\infty) - N[R'(c^\infty)N]^{-1}R'(c^\infty)f(c^\infty) \ \text{ on } \mathbb{R}_+, \ \ c^\infty(0) = c_0^\infty. \tag{8}$$

More precisely, we have

Theorem 8.2 *Consider the situation described in Theorem 8.1, let $c_0 \gg 0$ and $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ satisfy (6) and (7). Then*

(a) *Initial value problem (5) has a unique global solution $c^k(\cdot)$ for every $k \geq 0$. This solution is strictly positive on $\mathbb{R}_+$.*

(b) *The limiting equation (8) has a unique global solution $c^\infty(\cdot)$. This solution is strictly positive and remains in the manifold $\mathcal{M} = \{c \gg 0 : R(c) = 0\}$. If F_∞ denotes the right-hand side in (8), then $F_\infty(c) = (I - P(c))f(c)$ on $\mathcal{M}$, where*

$$P(c) = N[N^T C^{-1} N]^{-1} N^T C^{-1} \quad \textit{with } C = \textit{diag}(c_1, \dots, c_n)$$

is the projection onto S along $CS^\perp$.

(c) $c^k(t) \to c^\infty(t)$ *as* $k \to \infty$ *uniformly on compact subsets of* $(0,\infty)$*, where the initial value* c_0^∞ *in (8) is given as the unique positive equilibrium in the stoichiometric class* $c_0 + S$*.*

Proof. (a) Notice first that the fast reaction part $N\Lambda R(c)$ in (5) is quasi-positive by step 2 of the proof of Theorem 8.1. Hence the full right-hand side has this property and is locally Lipschitz, which implies local existence of a unique nonnegative solution $c(\cdot) = c^k(\cdot; c_0)$. Let $[0,T)$ be its maximal interval of existence and $\varphi(t) := \langle e, c(t)\rangle$ with e from (7). Evidently

$$\dot\varphi = \langle e, f(c)\rangle \le a(1 + \sum_{i=1}^n c_i) \le \rho a(1+\varphi) \quad \text{on } [0,T) \quad \text{with } \rho = \max\{1, 1/e_1, \ldots, 1/e_n\},$$

hence

$$|c(t)|_1 \le \rho\varphi(t) \le \psi(t) := \rho(1 + \langle e, c_0\rangle)e^{\rho a t} \quad \text{on } [0,T), \tag{9}$$

and therefore $T = \infty$. It remains to show $c(t) \gg 0$ on $[0,\tau]$ for arbitrary $\tau > 0$. Since the right-hand side $F(c) = f(c) + kN\Lambda R(c)$ in (5) is locally Lipschitz, it is Lipschitz of some constant L_k on the compact set $\{c \in \mathbb{R}^n_+ : |c|_1 \le \psi(\tau)\}$. Together with quasi-positivity of F this yields $F(c) \ge -L_k c$ for such c, hence $c_i(t) \ge c_{0,i}\, e^{-L_k t} > 0$ on $[0,\tau]$.
Consequently, initial value problem (5) has a unique global solution which stays strictly positive and satisfies (9) on all of $\mathbb{R}_+$.

(b) To prove the next part let us first show that the right-hand side F_∞ in (8) is well-defined on the manifold $\mathcal{M}$. Consider a fixed $c \in \mathcal{M}$, i.e. $c \gg 0$ and $c^{\nu_j} = 1/\kappa_j$ for $j = 1, \ldots, m$. Then

$$\frac{\partial R_j}{\partial c_k}(c) = \frac{\alpha_{j,k}}{c_k}c^{\alpha_j} - \kappa_j \frac{\beta_{j,k}}{c_k}c^{\beta_j} = -\frac{\nu_{j,k}}{c_k}c^{\alpha_j},$$

hence

$$R'(c) = -DN^TC^{-1} \quad \text{for } c \in \mathcal{M} \text{ with } D = \operatorname{diag}(c^{\alpha_1}, \ldots, c^{\alpha_m}),\ C = \operatorname{diag}(c_1, \ldots, c_n). \tag{10}$$

Therefore $R'(c)N = -DN^TC^{-1}N$ is negative definite, in particular invertible. Since $R'(c)N$ depends locally Lipschitz continuous on $c \in \mathcal{M}$, the inverse also has this property. Hence $F_\infty(c)$ is well-defined and locally Lipschitz continuous on $\mathcal{M}$. Moreover $F_\infty(c)$ has the representation mentioned in (b), and is tangent to $\mathcal{M}$ due to $R'(c)F_\infty(c) = 0$. As a consequence, the limiting equation (8) has a unique local solution $c^\infty(\cdot)$ by Corollary 2.1. This solution satisfies $|c^\infty(t)|_1 \le \psi(t)$ on its maximal interval of existence, where ψ is given in (9); notice that $\langle e, F_\infty(c)\rangle \le a(1+|c|_1)$ on $\mathcal{M}$ with e from (7). Therefore $c^\infty(\cdot)$ exists globally if it stays strictly positive. Assume, on the contrary, that there is $T > 0$ such that $c(t) \gg 0$ on $[0,T)$ and $\underline{\lim}_{t \nearrow T}\, c_i(t) = 0$ for some i.
We claim that there are $\delta, \mu > 0$ such that $c(t) = c_0^\infty \cdot e^{x(t)}$ with $x(t) \in K_{\delta,\mu}$, where

$$K_{\delta,\mu} = \{x \in S^\perp : x_i \le \mu \text{ for } i = 1, \ldots, m,\ \langle e^x, y\rangle \ge \delta \text{ for all } y \in S^\perp \cap \mathbb{R}^n_+, |y|_1 = 1\}. \tag{11}$$

If this holds we obviously arrive at a contradiction in case $K_{\delta,\mu}$ is bounded.

Of course $c(t) = c_0^\infty \cdot e^{x(t)}$ is valid with some function $x : [0,T) \to S^\perp$ by Theorem 8.1(a), since $R(c(t)) = 0$ on $[0,T)$. Moreover $c(t) \in [0,M]^n$ with $M = \psi(T)$ by the estimate given in (9), hence $x_i(t) \le \mu$ on $[0,T)$ for $i = 1,\dots,m$ with some $\mu > 0$. Let L be a Lipschitz constant of f on $[0,M]^n$, and recall that this implies $f(c) \ge -Lc$ on $[0,M]^n$ due to quasi-positivity of f. Fix $y \in S^\perp \cap \mathbb{R}_+^n$ with $|y|_1 = 1$. Then

$$\frac{d}{dt}\langle c(t), y\rangle = \langle f(c(t)), y\rangle \ge -L\langle c(t), y\rangle \quad \text{on } [0,T),$$

hence

$$\langle c(t), y\rangle = \langle c_0^\infty \cdot e^{x(t)}, y\rangle \ge \eta e^{-Lt} \text{ on } [0,T) \text{ with } \eta = \frac{1}{n}\min_i c_{0,i}^\infty.$$

This yields $\langle e^{x(t)}, y\rangle \ge \delta$ with $\delta = \eta e^{-LT}/\max_i c_{0,i}^\infty > 0$ and therefore $x(t) \in K_{\delta,\mu}$ on $[0,T)$.
It remains to show that $K_{\delta,\mu}$ is bounded. If not there is $(x^k) \subset K_{\delta,\mu}$ such that $|x^k|_1 \to \infty$, and w.l.o.g. $x_i^k \to -\infty$ if $i \in I$ and (x_i^k) is bounded if $i \notin I$ with some $\emptyset \ne I \subset \{1,\dots,n\}$. We may also assume $z^k := x^k/|x^k|_1 \to z$. Evidently $z \in S^\perp$ with $|z|_1 = 1$ and $z_i \le 0$ if $i \in I$, $z_i = 0$ if $i \notin I$. Hence $y := -z$ is "admissible" in the definition of $K_{\delta,\mu}$ which leads to the contradiction

$$0 < \delta \le \langle e^{x^k}, y\rangle = \sum_{i\in I} y_i e^{x_i^k} \le \sum_{i\in I} e^{x_i^k} \to 0.$$

(c) The following facts will be used frequently below. Let $J = [0,T]$ with $T > 0$, $c(\cdot)$ be a solution of (5) for arbitrary $k \ge 0$ and $c^*(t)$ denote the unique equilibrium of (1) in the class $c(t) + S$. Then there are $K, L, M, M^*, \eta > 0$, all depending on T but independent of $k \ge 0$, such that

$$\begin{aligned} &0 < c_i(t) \le M, \;\; 0 < \eta \le c_i^*(t) \le M^* \quad \text{on } J \text{ for } i = 1,\dots,n \\ &|f(c)|_\infty \le K \;\text{ and } f \text{ is Lipschitz of constant } L \text{ on } [0,M]^n. \end{aligned} \tag{12}$$

The bounds for $c^*(t)$ need further explanation, while the other facts are direct consequences of parts (a) and (b) of this proof. Theorem 8.1(c) yields $c^*(t) = \lim_{\tau\to\infty} z(\tau)$ where $\dot z(\tau) = N\Lambda R(z(\tau))$ on $\mathbb{R}_+$ and $z(0) = c(t)$. Fix $\overline{c} \gg 0$ with $R(\overline{c}) = 0$ and let V be given by (2). Now observe first that $h(x) = x\ln(x/\overline{x}) - (x - \overline{x})$ with $\overline{x} > 0$ has $h''(x) = 1/x$ for $x > 0$. Hence

$$h(x) = \int_{\overline{x}}^x \int_{\overline{x}}^s \frac{1}{r}\,dr\,ds \ge \frac{(x-\overline{x})^2}{2x} \ge \frac{x}{2} - \overline{x} \quad \text{for } x \ge \overline{x},$$

and therefore $x \le 2(\overline{x} + h(x))$ for all $x \ge 0$ which yields

$$|c|_1 = \sum_{i=1}^n c_i \le 2(|\overline{c}|_1 + V(c)) \quad \text{for all } c \in \mathbb{R}_+^n.$$

Consequently,

$$|c^*(t)|_1 \le 2(|\overline{c}|_1 + V(c^*(t))) \le 2(|\overline{c}|_1 + V(c(t))),$$

since V is a Lyapunov function for (1). This yields upper bounds on the $c_i^*(t)$. Having upper bounds, the same arguments as given below (11) show that $c^*(t) = c_0^\infty \cdot e^{x(t)}$ with $x(t) \in K_{\delta,\mu}$

on J, where $\delta, \mu > 0$ are independent of $k \geq 0$. Since $K_{\delta,\mu}$ from (11) is bounded, this also yields strictly positive lower bounds for the $c_i^*(t)$.

In the subsequent steps we omit the argument t whenever this is reasonable.

(i) Consider $V(c, \phi(c))$ where $V : (0,\infty)^n \times (0,\infty)^n \to \mathbb{R}_+$ is given by

$$V(c, c^*) = \sum_{i=1}^{n} c_i \ln \frac{c_i}{c_i^*} - (c_i - c_i^*), \tag{13}$$

and $\phi(c)$ denotes the unique equilibrium in the class $c + S$ for $c \gg 0$; recall that $\phi : (0,\infty)^n \to (0,\infty)^n$ is well defined by Theorem 8.1(a), and

$$\phi(c) = c^* \quad \text{iff} \quad R(c^*) = 0 \text{ and } Ec^* = Ec.$$

We claim that for every $T > 0$ there exist $\omega_T > 0$ and $M_T > 0$ such that

$$\frac{d}{dt} V(c^k(t), \phi(c^k(t))) \leq M_T - \omega_T \, k \sum_{j=1}^{m} \lambda_j R_j(c^k(t))^2 \quad \text{on } [0,T] \text{ for all } k \geq 0, \tag{14}$$

where $c^k(\cdot)$ is the solution of (5).

To establish (14), fix $k \geq 0$ and let $c(\cdot) = c^k(\cdot)$ as well as $c^*(\cdot) = \phi(c^k(\cdot))$. Evidently $c^*(\cdot)$ is differentiable if ϕ has this property. Since ϕ is implicitely defined by $F(c, c^*) = 0$ with $F(c, c^*) = (R(c^*), E(c^* - c))$, differentiability of ϕ follows by the implicit function theorem in case $\det\Big(\frac{\partial F}{\partial c^*}(c, c^*)\Big) \neq 0$ if $F(c, c^*) = 0$. Let $x \in \ker\Big(\frac{\partial F}{\partial c^*}(c, c^*)\Big)$ for such c, c^*. This implies $R'(c^*)x = 0$ and $Ex = 0$, hence $\frac{x}{c^*} \in S^\perp$ by (10) and $x \in S$. Therefore $x = 0$.

Consequently, to establish (14) we have to obtain appropriate bounds for

$$\frac{d}{dt} V(c, c^*) = \langle \ln \frac{c}{c^*}, f(c) \rangle + k \langle \ln \frac{c}{c^*}, N\Lambda R(c) \rangle + \langle \mathbf{1} - \frac{c}{c^*}, \dot{c}^* \rangle \quad \text{with } \mathbf{1} = (1, \ldots, 1). \tag{15}$$

The first term on the right is bounded due to (12) combined with

$$f_i(c) \ln c_i \leq -L c_i \ln c_i \leq L/e \quad \text{if } 0 < c_i \leq 1. \tag{16}$$

To obtain an estimate for the last term, notice that $R(c^*(t)) \equiv 0$ implies $R'(c^*)\dot{c}^* = 0$, hence $\dot{c}^*/c^* \in S^\perp$ by (10). On the other hand $E\dot{c}^* = E\dot{c} = Ef(c)$, i.e. $\dot{c}^* - f(c) \in S$ and therefore $\langle \dot{c}^*/c^*, \dot{c}^* - f(c) \rangle = 0$. This yields

$$\sum_{i=1}^{n} \frac{(\dot{c}_i^*)^2}{c_i^*} \leq \sum_{i=1}^{n} \frac{f_i(c)^2}{c_i^*}, \quad \text{hence} \quad |\dot{c}^*|_2 \leq \sqrt{M^*/\eta}\, |f(c)|_2.$$

By means of (12) it follows that $|\langle \mathbf{1} - \frac{c}{c^*}, \dot{c}^* \rangle|$ is bounded. Hence there is $M_T > 0$ such that

$$\frac{d}{dt} V(c, c^*) \leq M_T + k \langle \ln \frac{c}{c^*}, N\Lambda R(c) \rangle,$$

and then (14) follows by means of (4) and (12); recall that k_j in (4) corresponds to $k\lambda_j$.

(ii) Consider $W : (0,\infty)^n \to \mathbb{R}_+$ defined by

$$W(c) = |D(c)^{-1/2}\Lambda^{1/2}R(c)|_2 \quad \text{with } D(c) = \operatorname{diag}(c^{\alpha_1},\ldots,c^{\alpha_m}).$$

Suppose that $c_i^k(t) \geq \gamma > 0$ on $[0,a]$ for $i = 1,\ldots,n$ and all $k \geq 0$. We claim that, in this situation, there are constants $K_1, K_2, \sigma > 0$ and $k_0 > 0$ such that

$$W(c^k(t)) \leq W(c_0)K_1 e^{-\sigma k t} + \frac{K_2}{\sigma k} \quad \text{on } [0,a] \text{ for all } k \geq k_0, \tag{17}$$

where $c^k(\cdot)$ is the solution of (5). Let $\varphi(t) = W(c^k(t))$ on $[0,a]$. Then (17) is valid if the differential inequality

$$\varphi' \leq L_1 - k(2\sigma - L_2\varphi)\varphi \quad \text{a.e. on } [0,a] \quad \text{for all } k \geq k_0, \tag{18}$$

with $L_1, L_2, \sigma > 0$ independent of k, is satisfied. Indeed, (18) implies

$$\varphi(t) \leq \varphi(0)e^{-2\sigma k t}\exp\Big(kL_2\int_0^t \varphi(\tau)d\tau\Big) + L_1\int_0^t e^{-2\sigma k(t-s)}\exp\Big(kL_2\int_s^t \varphi(\tau)d\tau\Big)ds$$

by Gronwall's lemma, and

$$kL_2\int_s^t \varphi(\tau)d\tau \leq kL_2\sqrt{t-s}\Big(\int_s^t \varphi(\tau)^2 d\tau\Big)^{1/2} \leq k\sigma(t-s) + k\frac{L_2^2}{4\sigma}\int_s^t \varphi(\tau)^2 d\tau.$$

By (12) there is $\rho > 0$ such that $\varphi(\tau)^2 = \langle D(c^k)^{-1}\Lambda R(c^k), R(c^k)\rangle \leq \rho\sum_{j=1}^m \lambda_j R_j(c^k)^2$, hence the integrated version of (14) implies

$$k\frac{L_2^2}{4\sigma}\int_s^t \varphi(\tau)^2 d\tau \leq \rho\frac{L_2^2}{4\sigma}\int_s^t k\sum_{j=1}^m \lambda_j R_j(c^k(\tau))^2 d\tau \leq M_{a,\gamma} \quad \text{for } 0 \leq s \leq t \leq a,$$

with some $M_{a,\gamma}$ independent of $k \geq 0$. Consequently

$$\varphi(t) \leq \varphi(0)e^{M_{a,\gamma}}e^{-\sigma k t} + L_1 e^{M_{a,\gamma}}\int_0^t e^{-\sigma k s}ds,$$

hence (17) is valid with $K_1 = M_{a,\gamma}$ and $K_2 = L_1 e^{M_{a,\gamma}}$.

It remains to establish (18), where we consider $\Psi(c) = \frac{1}{2}W(c)^2 = \frac{1}{2}\langle D(c)^{-1}\Lambda R(c), R(c)\rangle$ to keep the computations shorter. It follows by elementary calculations that

$$\nabla\Psi(c) = -\frac{1}{2}C^{-1}A^T D(c)^{-1}\Lambda R(c)^2 + R'(c)^T D(c)^{-1}\Lambda R(c)$$

with $C = \operatorname{diag}(c_1,\ldots,c_n)$ and $A = (\alpha_{j,k})$; notice that $\dfrac{\partial c^{\alpha_j}}{\partial c_k} = c^{\alpha_j}\alpha_{j,k}\dfrac{1}{c_k}$ and recall that $R(c)^2$ is short for $(R_1(c)^2,\ldots,R_m(c)^2)$. Insertion of $\dot c = f(c) + kN\Lambda R(c)$ into $\langle\nabla\Psi(c),\dot c\rangle$ leads to four different terms that are estimated below; recall that we only need estimates that are valid for $c \in [\gamma, M]^n$. For the first term we obtain

$$-\frac{1}{2}\langle C^{-1}A^T D(c)^{-1}\Lambda R(c)^2, f(c)\rangle \leq |\frac{1}{2}D(c)^{-1}\Lambda R(c)^2|_1 \, |AC^{-1}f(c)|_\infty \leq l_1\Psi(c),$$

while for the second one we have

$$-\frac{1}{2}\langle C^{-1}A^T D(c)^{-1}\Lambda R(c)^2, kN\Lambda R(c)\rangle \leq k|\frac{1}{2}D(c)^{-1}\Lambda R(c)^2|_1 \, |AC^{-1}N\Lambda R(c)|_\infty$$

$$\leq k\Psi(c)||AC^{-1}ND(c)^{1/2}\Lambda^{1/2}|| \cdot |D(c)^{-1/2}\Lambda^{1/2}R(c)|_\infty \leq l_2 k\Psi(c)^{3/2},$$

where $||\cdot||$ denotes an appropriate matrix norm. To obtain an upper bound for the third term notice that

$$\frac{\partial R_j}{\partial c_k}(c) = \frac{\alpha_{j,k}}{c_k}c^{\alpha_j} - \kappa_j\frac{\beta_{j,k}}{c_k}c^{\beta_j} = -\frac{\nu_{j,k}}{c_k}c^{\alpha_j} + \frac{\beta_{j,k}}{c_k}R_j(c),$$

i.e.

$$R'(c) = -D(c)N^TC^{-1} + \mathcal{R}(c)B^TC^{-1} \quad \text{with } \mathcal{R}(c) = \operatorname{diag}(R_1(c), \ldots, R_m(c)),\ B = (\beta_{j,k}).$$

Hence $R'(c)$ is bounded on $[\gamma, M]^n$ and therefore

$$\langle R'(c)^T D(c)^{-1}\Lambda R(c), f(c)\rangle = \langle D(c)^{-1/2}\Lambda^{1/2}R(c), D(c)^{-1/2}\Lambda^{1/2}R'(c)f(c)\rangle \leq l_3\Psi(c)^{1/2}.$$

Concerning the last term, above formula for $R'(c)$ yields

$$\langle R'(c)^T D(c)^{-1}\Lambda R(c), kN\Lambda R(c)\rangle =$$

$$-k\langle N\Lambda R(c), C^{-1}N\Lambda R(c)\rangle + k\langle \Lambda R(c)^2, D(c)^{-1}B^TC^{-1}N\Lambda R(c)\rangle.$$

Evidently

$$\langle N\Lambda R(c), C^{-1}N\Lambda R(c)\rangle \geq \frac{1}{M}\langle N^TN\Lambda R(c), \Lambda R(c)\rangle \geq \frac{\mu}{M}|\Lambda R(c)|_2^2,$$

where $\mu > 0$ is the smallest eigenvalue of the positive definite matrix N^TN. Therefore

$$\langle N\Lambda R(c), C^{-1}N\Lambda R(c)\rangle \geq \sigma_0\Psi(c) \quad \text{with some } \sigma_0 > 0,$$

where σ_0 only depends on γ, M.
Since the remaining part can be estimated in the same manner as the second term, we obtain

$$\langle R'(c)^T D(c)^{-1}\Lambda R(c), kN\Lambda R(c)\rangle \leq -k\sigma_0\Psi(c) + l_4 k\Psi(c)^{3/2}.$$

Altogether these estimates imply

$$\frac{d}{dt}\Psi(c) \leq l_1\Psi(c) + l_3\Psi(c)^{1/2} + k(l_2 + l_4)\Psi(c)^{3/2} - k\sigma_0\Psi(c)$$

along the solutions $c(\cdot)^k$ of (5). By means of $\frac{d}{dt}\Psi(c^k(t)) = \varphi(t)\varphi'(t)$ a.e. and $\varphi' = 0$ a.e. on $\{t \in [0,a] : \varphi(t) = 0\}$ it follows that

$$\varphi' \le L_1 + kL_2\varphi^2 + \frac{l_1 - k\sigma_0}{2}\varphi \quad \text{a.e on } [0,a]$$

with certain $L_1, L_2 > 0$ that are independent of k. Therefore (18) holds with $\sigma = \sigma_0/8$ and $k_0 = 2l_1/\sigma_0$, say.

(iii) Let $a > 0$ and suppose that $\min_i c_i^k(t) \ge \gamma$ on $[0,a]$ for all large k with some $\gamma > 0$. In this situation (17) is valid, and we claim that this implies $c^k(t) \to c^\infty(t)$ as $k \to \infty$, uniformly on compact subsets of $(0,a]$. Since the solution $c^\infty(\cdot)$ of (8) is unique, it suffices to show that, given any sequence $k_l \to \infty$, there is a subsequence of (c^{k_l}) which converges to c^∞, locally uniformly on $(0,a]$. Keeping this in mind, we again write c^k instead of c^{k_l} and subsequences thereof.

By (17) and (12) there exist $L_1, L_2, \sigma > 0$ (depending on a, γ but independent of k) and $k_0 > 0$ such that

$$|R(c^k(t))|_2 \le L_1 e^{-\sigma k t} + L_2/k \quad \text{on } [0,a] \ \text{ for all } k \ge k_0. \tag{19}$$

Hence, given $\epsilon \in (0,a)$, $kR(c^k(t))$ is bounded on $[\epsilon, a]$ uniformly with respect to $k \ge 0$. Exploitation of the differential equation in (5) shows that (c^k) is relatively compact in $C([\epsilon,a];\mathbb{R}^n)$ for all $\epsilon \in (0,a)$. Consideration of $\epsilon_l \searrow 0$ together with the usual diagonalization procedure yields a subsequence of (c^k), again denoted by (c^k), such that $c^k(t) \to c(t)$ locally uniformly on $(0,a]$. Since $(kR(c^k(\cdot)))$ is weakly relatively compact in $L^1([0,a];\mathbb{R}^n)$ by (19), we may also assume $k\Lambda R(c^k) \rightharpoonup \phi$ in $L^1([0,a];\mathbb{R}^n)$. Therefore integration of the differential equation for c^k from s to t and $k \to \infty$ yields

$$c(t) = c(s) + \int_s^t f(c(\tau))d\tau + N\int_s^t \phi(\tau)d\tau \quad \text{for } 0 < s < t \le a,$$

hence the limit $c(\cdot)$ is absolutely continuous and satisfies

$$\dot c = f(c) + N\phi(t) \quad \text{a.e. on } [0,a].$$

Evidently $c(\cdot)$ also satisfies $R(c(t)) = 0$ and $c(t) \in [\gamma, M]^n$ on $(0,a]$. Consequently

$$0 = R'(c(t))\dot c(t) = R'(c(t))[f(c(t)) + N\phi(t)] \quad \text{a.e. on } [0,a],$$

which implies

$$\phi(t) = -[R'(c(t))N]^{-1}R'(c(t))f(c(t)) \quad \text{a.e. on } [0,a];$$

recall from step (b) that $R'(c)N$ is invertible on $\{c \gg 0 : R(c) = 0\}$. Hence $c(\cdot)$ is a solution of the differential equation in (8) and the limit $c(0+) = \lim_{t\to 0+} c(t)$ exists since the right-hand side $F_\infty(c)$ is bounded on $[\gamma, M]^n \cap \mathcal{M}$. Moreover

$$Ec(t) = \lim_{k\to\infty} Ec^k(t) = Ec_0 + \lim_{k\to\infty}\int_0^t Ef(c^k(s))ds \ \text{ on } (0,a]$$

implies $|Ec(t) - Ec_0| \leq ||E||Kt$, hence $Ec(0+) = Ec_0$. Evidently $R(c(0+)) = 0$ and therefore $c(0+)$ is the unique equilibrium in the class $c_0 + S$, i.e. $c(0+) = c_0^\infty$. This means $c(t) = c^\infty(t)$ on $[0, a]$, hence the claim is proved.

(iv) To finish the proof of (c) let us first show that c^k converges to c^∞, locally uniformly on some small interval $(0, a]$. This holds by the previous step if there are $a, \gamma > 0$ such that $\min_i c_i^k(t) \geq \gamma$ on $[0, a]$ for all large $k \geq 0$. Let $z(\cdot)$ be the solution of

$$\dot{z} = N\Lambda R(z) \;\text{ on } \mathbb{R}_+, \;\; z(0) = c_0.$$

By Theorem 8.1(c) there is $\eta > 0$ such that $z_i(t) \geq 2\eta$ on $\mathbb{R}_+$ for all i, and $z(t) \to \overline{c}$ as $t \to \infty$, where $\overline{c}$ is the equilibrium in the class $c_0 + S$ (i.e. $\overline{c} = c_0^\infty$). Let V denote the Lyapunov function from (2), and $\delta := \frac{1}{4} \min_l \overline{c}_l$. Then $V(z(t)) \searrow 0$ as $t \to \infty$ implies $V(z(T)) \leq \delta$ for some $T > 0$. Consider $z^k(t) = c^k(t/k)$ on $[0, T]$ and notice that z^k is the solution of

$$\dot{z}^k = \frac{1}{k} f(z^k) + N\Lambda R(z^k) \;\text{ on } [0, T], \;\; z^k(0) = c_0.$$

Due to the continuous dependence of $z^k(\cdot)$ on the right-hand side it follows that $z^k \to z$ in $C([0, T]; \mathbb{R}^n)$, hence $\min_i z_i^k(t) \geq \eta$ on $[0, T]$ and $V(z^k(T)) \leq 2\delta$ for all large k. This means

$$\min_i c_i^k(t) \geq \eta > 0 \text{ on } [0, T/k] \quad \text{and} \quad V(c^k(T/k)) \leq 2\delta \text{ for all large } k.$$

Since $V(c)$ has $\dot{V}(c) = \langle \ln(c/\overline{c}), f(c) \rangle$, the uniform bounds given in (12) together with (16) imply $\frac{d}{dt} V(c^k(t)) \leq M_1$ on $[0, 1]$, say, for all $k \geq 0$ with some $M_1 > 0$. Hence there is $a > 0$ such that $V(c^k(t)) \leq 3\delta$ on $[T/k, a]$ for all $k \geq k_0$. Therefore $c := c^k(t)$ with $k \geq k_0$ and $t \in [T/k, a]$ satisfies

$$\frac{c_i}{\overline{c}_i} \ln \frac{c_i}{\overline{c}_i} + \frac{1}{4} \leq \frac{c_i}{\overline{c}_i} \quad \text{for } i = 1, \ldots, n,$$

which implies $c_i^k(t) \geq \rho \overline{c}_i$ on $[T/k, a]$ where $\rho > 0$ is the smallest solution of $r \ln r + 1/4 = r$. Hence

$$\min_i c_i^k(t) \geq \gamma := \min\{4\delta\rho, \eta\} > 0 \;\text{ on } [0, a] \text{ for } k \geq k_0.$$

Consequently,

$$T := \sup\{a > 0 : c^k(t) \to c^\infty(t) \text{ locally uniformly on } (0, a]\} > 0$$

by step (iii), and it remains to show $T = \infty$. Assume that $T < \infty$ and let

$$\delta = \frac{1}{4} \min\{c_i^\infty(t) : t \in [0, T], i = 1, \ldots, n\} > 0.$$

Since the estimates of type (12) are valid for the fixed interval $[0, T+1]$, there is $M_2 > 0$ such that

$$\langle \ln \frac{c^k(t)}{\overline{c}}, f(c^k(t)) \rangle \leq M_2 \;\text{ on } [0, T+1] \text{ for all } k \geq 0 \text{ and all } \overline{c} \in [\delta, M]^n.$$

Let $0 < a < \min\{\frac{\delta}{2M_2}, 1, T\}$. Evidently $c^k(T-a) \to \overline{c} := c^\infty(T-a) \in [\delta, M]^n$. Consider V from (2) with this particular $\overline{c}$. Then

$$\frac{d}{dt}V(c^k(t)) \leq M_2 \text{ on } [0, T+1] \quad \text{and} \quad V(c^k(T-a)) \leq \delta \text{ for all } k \geq k_0,$$

hence $V(c^k(t)) \leq 2\delta$ on $[T-a, T+a]$ for all $k \geq k_0$. Therefore, by a repetition of the arguments given above, there is $\gamma > 0$ such that $\min_i c_i^k(t) \geq \gamma$ on $[0, T+a]$ for all large k. This implies $c^k(t) \to c^\infty(t)$ locally uniformly on $(0, T+a]$ by step (iii), a contradiction. Hence $T = \infty$ which ends the proof. □

8.3 Reactions of diffusive species

In this final section we consider a single reversible reaction $A + B \rightleftharpoons P$ between mobile species inside an isolated vessel, which leads to the following reaction diffusion system where u, v and w denote the concentrations of A, B and P, respectively.

$$\begin{aligned}
&u_t = d_1\Delta u - k(uv - \kappa w) && \text{in } (0,\infty)\times\Omega \\
&v_t = d_2\Delta v - k(uv - \kappa w) && \text{in } (0,\infty)\times\Omega \\
&w_t = d_3\Delta w + k(uv - \kappa w) && \text{in } (0,\infty)\times\Omega \\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 && \text{on } (0,\infty)\times\Gamma \\
&u(0,\cdot) = u_0,\ v(0,\cdot) = v_0,\ w(0,\cdot) = w_0 && \text{in } \Omega
\end{aligned}$$

with diffusion coefficients $d_k > 0$, rate constant $k > 0$ and $\kappa > 0$, where we also assume that $\Omega \subset \mathbb{R}^n$ is open bounded with smooth boundary Γ.

It is known that this system has a unique classical solution if the initial values belong to $L^\infty(\Omega; \mathbb{R}_+)$; see Morgan [86] and remember Remark 6.3. Here (u, v, w) is said to be a classical solution on $[0, T]$ if $u, v, w \in C^{1,2}((0,T)\times\Omega) \cap C((0,T)\times\overline{\Omega})$ with $u_{x_i}, v_{x_i}, w_{x_i} \in C((0,T)\times\overline{\Omega})$ such that the differential equation holds in $(0,T)\times\Omega$, the boundary condition is valid on $(0,T)\times\Gamma$, and the initial value is attained in $L^p(\Omega)$ (e.g. $|u(t,\cdot) - u_0|_p \to 0$ as $t \to 0+$) for every $p \geq 1$.

As before we are interested in the limiting case $k \to \infty$ of an instantaneous reaction. Our subsequent result is only a starting point, since we have to assume equal diffusion coefficients so far. In this case we may assume $d_k = 1$ for $k = 1, 2, 3$, hence we consider the system

$$\frac{dc}{dt} = \Delta c - k\nu r(c) \ \text{ in } (0,\infty), \quad \frac{\partial c}{\partial \nu} = 0 \ \text{ on } (0,\infty)\times\Gamma, \quad c(0,\cdot) = c_0 \ \text{ in } \Omega \tag{20}$$

for $c = (u, v, w)$ with $\nu = (-1, -1, 1)^T$ (notice that here ν is used with a different meaning, but a confusion with the outer normal is not possible) and $r(c) = uv - \kappa w$. Let us note in passing that in this case and for initial value in $L^\infty(\Omega; \mathbb{R}^3_+)$ the set

$$K = \{(u, v, w) \in L^\infty(\Omega; \mathbb{R}^3_+) : u + w \leq |u_0|_\infty + |w_0|_\infty,\ v + w \leq |v_0|_\infty + |w_0|_\infty \text{ a.e. in } \Omega\}$$

is positively invariant for (20), hence global existence of (mild) solutions for (20) in $L^p(\Omega)$ (for every $p \geq 1$) follows immediately, and this mild solution is in fact a classical solution in the sense described above, which follows by standard techniques for semilinear evolution equations with analytic semigroup.

A formal application of Theorem 8.2 (with $f(c) = \Delta c$) suggests that the limiting problem is given by

$$\frac{dc}{dt} = (I - P(c))\Delta c \ \text{ in } (0,\infty), \quad \frac{\partial c}{\partial \nu} = 0 \ \text{ on } (0,\infty) \times \Gamma, \ \ c(0,\cdot) = c_0^\infty \ \text{ in } \Omega, \tag{21}$$

where the projections $P(c)$ are given as

$$P(c) = \frac{\nu \otimes r'(c)}{\langle \nu, r'(c) \rangle} = \frac{1}{u+v+\kappa} \begin{pmatrix} v & u & -\kappa \\ v & u & -\kappa \\ -v & -u & \kappa \end{pmatrix},$$

and the initial value $c_0^\infty = (u_0^\infty, v_0^\infty, w_0^\infty)$ is determind by

$$u_0^\infty + w_0^\infty = u_0 + w_0, \quad v_0^\infty + w_0^\infty = v_0 + w_0, \quad u_0^\infty v_0^\infty = \kappa w_0^\infty. \tag{22}$$

The following result validates this expectation.

Theorem 8.3 *Let $\Omega \subset \mathbb{R}^n$ be open bounded with smooth boundary Γ. Let $c_0 = (u_0, v_0, w_0) \in L^\infty(\Omega; \mathring{\mathbb{R}}^3_+)$ and let $c^k(\cdot)$ denote the classical solution of (20) for $k > 0$ on $J = [0,T]$ with arbitrary $T > 0$. Then*

$$c_i^k \to c_i^\infty \ \text{ in } W^{s,1}(J; L^1(\Omega)) \cap L^1(J; W^{2s,1}(\Omega)) \ \text{ as } k \to \infty \text{ for } i = 1,2,3 \text{ and all } s < 1,$$

where c^∞ is the unique classical solution of (21) on J with initial value given by (22).

Proof. Let us first show that $(kr(c^k))_{k>0}$ is bounded in $L^1(Q_T)$ with $Q_T = (0,T) \times \Omega$. For this purpose let $R(c) = \int_\Omega |uv - \kappa w|\, dx$ for $c = (u,v,w) \in L^2(\Omega; \mathbb{R}^3)$ and consider the classical solution $c(\cdot) = c^k(\cdot)$ of (20) for fixed $k > 0$. Then $R(c(\cdot))$ is absolutely continuous with

$$\frac{d}{dt} R(c) = \int_\Omega (uv - \kappa w)_t \operatorname{sgn}(uv - \kappa w)\, dx$$

$$= -k \int_\Omega (u+v+\kappa)|uv - \kappa w|\, dx + \int_\Omega (v\Delta u + u\Delta v - \kappa \Delta w) \operatorname{sgn}(uv - \kappa w)\, dx \ \text{ a.e. on } J.$$

Since $\Delta(uv - \kappa w) = v\Delta u + u \Delta v - \kappa \Delta w + 2\langle \nabla u, \nabla v\rangle$ and $-\Delta$ is s-accretive in $L^1(\Omega)$, it follows that

$$\frac{d}{dt} R(c) \leq 2|\nabla u|_2 |\nabla v|_2 - k\kappa R(c) \ \text{ a.e. on } J,$$

hence

$$|kr(c^k)|_{L^1(Q_T)} = \int_0^T kR(c^k(t))\, dt \leq |r(c_0)|_{L^1(\Omega)} + |\nabla u^k|^2_{L^2(Q_T)} + |\nabla v^k|^2_{L^2(Q_T)}.$$

To establish appropriate bounds for the gradients, let $u^*, v^*, w^* \in (0,\infty)$ be such that $u^*v^* = \kappa w^*$, i.e. (u^*, v^*, w^*) is a positive equilibrium of the ordinary differential equation associated with (20). Consider

$$V(c) = \int_\Omega \Big(\phi_1(u) + \phi_2(v) + \phi_3(w)\Big)\,dx \quad \text{for } c = (u,v,w) \in L^2(\Omega;\mathbb{R}^3_+),$$

where $\phi(s) = \ln(s/s^*) - (s - s^*)$ for $s = u, v, w$; compare (2) in §8.1 above, and notice that $u(t,x), v(t,x), w(t,x) > 0$ on $(0,T] \times \overline{\Omega}$ since for instance $u_t \geq \Delta u - kLu$ in Q_T with $L = |u_0|_\infty + |w_0|_\infty$ and $u_0(x) > 0$ a.e. in Ω. Then

$$\frac{d}{dt}V(c) = \int_\Omega \Big(u_t \ln\frac{u}{u^*} + v_t \ln\frac{v}{v^*} + w_t \ln\frac{w}{w^*}\Big)\,dx$$

$$= -\int_\Omega \Big(\frac{|\nabla u|^2}{u} + \frac{|\nabla v|^2}{v} + \frac{|\nabla w|^2}{w}\Big)\,dx - k\int_\Omega (uv - \kappa w)(\ln(uv) - \ln(\kappa w))\,dx.$$

Consequently

$$\int_{Q_T} \Big(\frac{|\nabla u|^2}{u} + \frac{|\nabla v|^2}{v} + \frac{|\nabla w|^2}{w}\Big)\,dxdt \leq V(c_0)$$

and therefore $(\nabla u^k)_{k>0}$, $(\nabla v^k)_{k>0}$, $(\nabla w^k)_{k>0}$ are bounded in $L^2(Q_T)$ since u^k, v^k, w^k are bounded by $\max\{|u_0|_\infty + |w_0|_\infty, |v_0|_\infty + |w_0|_\infty\}$.
Hence $|kr(c^k)|_{L^1(Q_T)} \leq M$ with some $M = M(c_0) > 0$ for all $k > 0$, i.e. the inhomogenity in (20) is bounded in $L^1(Q_T;\mathbb{R}^3)$ uniformly with respect to $k > 0$. In this situation it follows from standard regularity results for inhomogeneous linear evolution equations with analytic semigroups that

$$(u^k)_{k>0},\ (v^k)_{k>0},\ (w^k)_{k>0} \ \text{ are bounded in } W^{s,1}(J;L^1(\Omega)) \cap L^1(J;W^{2s,1}(\Omega)) \text{ for all } s < 1.$$

Hence the components of $(c^k)_{k>0}$ are also relatively compact in the same space due to compact embedding of $W^{s',1}$ in $W^{s,1}$ for $s < s'$. Consequently, given any sequence $k_j \to \infty$, there is a subsequence $(c^{k_{j_l}})$ of (c^{k_j}), denoted by (c^{k_l}) for simplicity, such that

$$c_i^{k_l} \to c_i^\infty \ \text{ in } W^{s,1}(J;L^1(\Omega)) \cap L^1(J;W^{2s,1}(\Omega)) \ \text{ as } l \to \infty \text{ for } i = 1,2,3 \text{ and } s < 1.$$

To finish the proof it suffices to show that c^∞ is uniquely determined as the classical solution of (21) with u_0^∞ from (22); notice that the whole sequence converges to c^∞ then. For this purpose let $y^l = u^{k_l} + w^{k_l}$ and $z^l = v^{k_l} + w^{k_l}$. Evidently y^l and z^l are independent of l and given as $y^l = y$, $z^l = z$ where y, z are the classical solutions of

$$\begin{aligned} &y_t = \Delta y \ \text{ in } (0,T)\times\Omega, \quad \partial_\nu y = 0 \ \text{ on } (0,T)\times\Gamma, \quad y(0,\cdot) = u_0 + w_0 \ \text{ in } \Omega \\ &z_t = \Delta z \ \text{ in } (0,T)\times\Omega, \quad \partial_\nu z = 0 \ \text{ on } (0,T)\times\Gamma, \quad z(0,\cdot) = v_0 + w_0 \ \text{ in } \Omega. \end{aligned} \tag{23}$$

Since $r(c^k) \to 0$ in $L^1(Q_T)$ it follows that $c^\infty = (u^\infty, v^\infty, w^\infty)$ satisfies

$$u^\infty + w^\infty = y, \quad v^\infty + w^\infty = z, \quad u^\infty v^\infty = \kappa w^\infty \quad \text{and} \quad u^\infty, v^\infty, w^\infty \geq 0 \ \text{ a.e. in } Q_T. \tag{24}$$

Now a straight forward computation shows that the system of equations above has a unique nonnegative solution $(u^\infty, v^\infty, w^\infty)$ which is given by

$$\begin{aligned} u^\infty = \phi(y,z) + y, \;\; v^\infty = \phi(y,z) + z, \;\; w^\infty = -\phi(y,z) \\ \text{with} \quad \phi(y,z) = \phi(z,y) = \frac{1}{2}(\kappa^2 + 2\kappa(y+z) + (y-z)^2)^{1/2}. \end{aligned} \tag{25}$$

Due to this explicit representation it follows that u^∞, v^∞ and w^∞ have the regularity properites required for classical solutions. Furthermore

$$u_t^\infty + w_t^\infty = \Delta u^\infty + \Delta w^\infty, \;\; v_t^\infty + w_t^\infty = \Delta v^\infty + \Delta w^\infty, \;\; u_t^\infty v^\infty + u^\infty v_t^\infty = \kappa w_t^\infty,$$

and then a simple calculation shows that $c^\infty = (u^\infty, v^\infty, w^\infty)$ is a classical solution of the differential equation in (21). Due to (23), (24) and (25) it is also clear that $(u_0^\infty, v_0^\infty, w_0^\infty) := \lim_{t\to 0+} c^\infty(t)$ exists in $L^p(\Omega)$ for $p \geq 1$ and is characterized by (22). Hence c^∞ is a classical solution of (21), (22) on J.

Finally, let $c = (u,v,w)$ be any classical solution of (21) on J with initial value c_0^∞ from (22). A simple computation yields $(uv)_t = \kappa w_t$ in Q_T, hence also $uv = \kappa w$ in Q_T. Moreover, $y = u + w$ and $z = v + w$ are classical solutions of (23), hence c is uniquely determined by (24), i.e. $c = c^\infty$. □

The convergence in Theorem 8.3 is optimal in the sense that $s = 1$ is not admissible for initial values with $u_0 v_0 \neq \kappa w_0$; observe that convergence with $s = 1$ would imply convergence in $C(J; L^1(\Omega))$, but a jump at $t = 0$ develops as $k \to \infty$.

8.4 Remarks

Remark 8.1 Parts (a) and (c) of Theorem 8.1 are essentially contained in Horn/Jackson [70]. The proof of part (a) above is based on arguments from Feinberg [52], where existence of a unique equilibrium in each stoichiometric class is obtained for considerably more general systems of chemical reactions with mass-action kinetics. The argument given here to obtain strict positivity of solutions is taken from Vol'pert [109].

Remark 8.2 In Chapter 12.5 of Vol'pert/Hudjaev [110] the authors study the instantaneous reaction limit for systems of chemical reactions composed of slow and fast reactions under the assumption of mass-action kinetics but without macroscopic convection. This leads to initial value problems of type (5) with f having a special structure since it corresponds solely to additional slow reactions. In this setting convergence of solutions as $k \to \infty$ to the solution of a certain limiting problem is claimed in Theorem 12.5.1. There the basic idea is to rewrite the system (5) in terms of slow and fast variables and to apply classical singular perturbation theory, in particular Tihonov's theorem. Translated to our notation this means to apply the transformation $u = Ec$, $v = N^T c$, and to rewrite (5) in terms of u and v. If ϕ denotes the inverse transformation, this leads to

$$\begin{aligned} \dot{u} &= Ef(\phi(u,v)) && \text{on } \mathbb{R}_+, \quad u(0) = Ec_0 \\ \epsilon \dot{v} &= \epsilon N^T f(\phi(u,v)) + N^T N \Lambda R(\phi(u,v)) && \text{on } \mathbb{R}_+, \quad v(0) = N^T c_0 \end{aligned} \tag{26}$$

with $\epsilon = 1/k$. Then the limit problem is given by

$$\begin{aligned} \dot{u} &= Ef(\phi(u,v)) \quad \text{on } \mathbb{R}_+, \quad u(0) = Ec_0 \\ 0 &= R(\phi(u,v)). \end{aligned} \tag{27}$$

The result mentioned above states that (27) has a unique local solution (u^0, v^0) on some interval $(0,T)$ and that the solutions (u^ϵ, v^ϵ) converge to (u^0, v^0) as $\epsilon \to 0+$, uniformly on compact subsets of $(0,T)$. Unfortunately, a rigorous proof is not given. In particular, the Theorem of Tihonov, at least in the version given in [110], does not cover this situation since ϵ also appears on the right-hand side in (26). Furthermore, observe that (27) is an implicit formulation of the limit problem and requires knowledge of a complete set of conserved quantities, while the explicit formulation by means of (8) only involves the stoichiometric coefficients.

Let us also note that the direct proof of Theorem 8.2 given here yields a thorough understanding of the particular features of the system (5), which is indispensable for extensions to reaction-diffusion systems.

Some references to related papers in the engineering literature can be found in Erdi/Toth [49] and in Segel/Slemrod [102].

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